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STEADY-STATE THERMOELASTICITY FOR INITIALLY STRESSED BODIES

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CONTENTS

An elastic body, deformed from a state of zero stress and strain and uniform temperature by a large deformation and steady-state temperature distribution, is subsequently subjected to small displacements and steady-state temperature distributions. After a general analysis of the problem the work is specialized to the case when the initial large deformation is homogeneous at constant temperature. A general solution of the equations for the small superposed deformation and steady-state temperature distribution is obtained in terms of three stress functions valid for some regions of space including the half space and thick uniform plate, when two perpendicular extension ratios of the initial homogeneous deformation are equal. Applications are made to problems of a plane circular (penny-shaped) crack in an infinite medium and to half-space problems.

1. Introduction

Thermoelasticity for classical infinitesimal elasticity has received much attention in recent years. Here we are concerned with an ideally elastic body, deformed from a state of zero stress and strain and uniform temperature, which is subsequently subjected to small displacements and steady-state temperature distributions. The body is assumed to be isotropic initially but no further restriction is placed on the strain-energy function. There is, however, no difficulty in extending the analysis to apply to aeolotropic bodies. The work is an extension of that given by Green, Rivlin & Shield (1952) and Green & Zerna (1954) who considered small deformations superposed on a large deformation, all at constant temperature. A summary of basic formulae is given in § 2 and the general analysis of the problem is contained in § 3. In § 4 the special case of an initial large homogeneous deformation at constant temperature is considered and this is further specialized in the rest of the paper to the case when two extension ratios parallel to two rectangular Cartesian co-ordinate axes are equal. In § 5 a general solution of the equations for the small superposed deformation and steady-state temperature distribution in a compressible body is obtained in terms of three stress functions, valid for some regions of space including the half space and thick uniform plate.

The corresponding results for an incompressible body are given in § 6. The remaining sections of the paper are concerned with some applications of the theory to problems of a plane circular (penny-shaped) crack in an infinite medium and to half-space problems.

2. Notation and formulae

We use the notation and formulae given by Green & Adkins (1960) and we quote the main results here. Points in the body are defined by a general curvilinear system of co-ordinates θ_i which moves with the body as it deforms. The initial base vectors and metric tensors are $\mathbf{g}_i$, $\mathbf{g}^i$ and g_{ij}, g^{ij} respectively and the corresponding values in the deformed body are $\mathbf{G}_i$, $\mathbf{G}^i$ and G_{ij}, G^{ij} so that

$$\left.\begin{array}{c} g_{ij} = \mathbf{g}_i . \mathbf{g}_j, \qquad g^{ij} = \mathbf{g}^i . \mathbf{g}^j, \\ G_{ij} = \mathbf{G}_i . \mathbf{G}_j, \qquad G^{ij} = \mathbf{G}^i . \mathbf{G}^j, \\ g^{ir} g_{rk} = G^{ir} G_{rk} = \delta^i_k. \end{array}\right\} \tag{2·1}$$

The strain tensor is defined by $$\gamma_{ij} = \tfrac{1}{2}(G_{ij} - g_{ij}) \tag{2·2}$$
and we define γ^i_j by the formula

$$\gamma^i_j = g^{ir}\gamma_{rj} = \tfrac{1}{2}(g^{ir}G_{rj} - \delta^i_j). \tag{2·3}$$

Latin indices take the values 1, 2, 3 and repeated indices are summed.

The contravariant stress tensor τ^{ij}, measured per unit area of the deformed body, and referred to θ_i co-ordinates in the deformed body, is given by

$$\tau^{ij} = \Phi g^{ij} + \Psi B^{ij} + pG^{ij} \tag{2·4}$$

if the body is initially isotropic and homogeneous, where

$$B^{ij} = (g^{ij}g^{rs} - g^{ir}g^{js})\, G_{rs} \tag{2·5}$$

and
$$\Phi = \frac{2}{\sqrt{I_3}}\frac{\partial W}{\partial I_1}, \quad \Psi = \frac{2}{\sqrt{I_3}}\frac{\partial W}{\partial I_2}, \quad p = 2\sqrt{I_3}\frac{\partial W}{\partial I_3}. \tag{2·6}$$

In (2·6) W is the Helmholtz free-energy function

$$W = W(I_1, I_2, I_3, T), \tag{2·7}$$

where T is temperature and I_i are the three strain invariants

$$I_1 = g^{ij}G_{ij}, \quad I_2 = I_3 G^{ij} g_{ij}, \quad I_3 = G/g, \tag{2·8}$$

and
$$G = |G_{ij}|, \quad g = |g_{ij}|. \tag{2·9}$$

If the body is incompressible then

$$I_3 = G/g = 1, \quad W = W(I_1, I_2, T), \tag{2·10}$$

and
$$\Phi = 2\frac{\partial W}{\partial I_1}, \quad \Psi = 2\frac{\partial W}{\partial I_2}. \tag{2·11}$$

The stress tensor is still given by (2·4) but p is now an arbitrary scalar representing a hydrostatic tension.

When body forces are zero the equations of equilibrium are

$$\partial \mathbf{T}_i/\partial\theta^i \equiv \mathbf{T}_{i,i} = \mathbf{0}, \tag{2·12}$$

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or
$$\tau^{ij}\|_i = 0, \tag{2·13}$$
where
$$\mathbf{T}_i = \sqrt{(G)}\, \tau^{ij} \mathbf{G}_j, \tag{2·14}$$
and ρ, the density of the deformed body, is related to the (constant) density ρ_0 of the undeformed body by the equation
$$\rho\sqrt{G} = \rho_0\sqrt{g}. \tag{2·15}$$
The double line in (2·13) denotes covariant differentiation with respect to the deformed body using the metric tensors G_{ij}, G^{ij} in forming the Christoffel symbols.

If $\mathbf{Q}$ denotes the heat-conduction vector per unit area of the deformed body and
$$\mathbf{Q} = Q^i \mathbf{G}_i, \tag{2·16}$$
then, for steady-state conditions,
$$Q^i\|_i = 0 \quad \text{or} \quad \frac{\partial}{\partial \theta^i}(Q^i\sqrt{G}) = 0. \tag{2·17}$$
When the body is initially isotropic
$$-Q^i = (\mathscr{C}_1 \delta^i_j + \mathscr{C}_2 \gamma^i_j + \mathscr{C}_3 \gamma^i_m \gamma^m_j)\, T\|^j + \epsilon^{ijk}(\mathscr{C}_4 \delta^s_j \delta^t_k + \mathscr{C}_5 \delta^s_j \gamma^t_k + \mathscr{C}_6 \gamma^s_j \gamma^t_k)\, \gamma^r_t\, T\|_r\, T\|_s, \tag{2·18}$$
where
$$\epsilon^{ijk} = e_{ijk}/\sqrt{G}, \quad T\|^j = G^{jr}\, T\|_r = G^{jr}\frac{\partial T}{\partial \theta^r}, \tag{2·19}$$
and e_{ijk} is the alternating Cartesian tensor. Also $\mathscr{C}_1, \ldots, \mathscr{C}_6$ are polynomials in the invariants
$$\left.\begin{aligned} &I_1, \quad I_2, \quad I_3, \\ &I_4 = T\|^i\, T\|_i, \\ &I_5 = T\|_i\, T\|^j \gamma^i_j, \\ &I_6 = T\|_i\, T\|^j \gamma^i_k \gamma^k_j, \end{aligned}\right\} \tag{2·20}$$
with coefficients which are continuous functions of T. The heat-conduction vector satisfies the condition
$$-Q^i\, T\|_i \geqslant 0. \tag{2·21}$$

3. Small deformations superposed on a large deformation

We consider a deformation of the body which is such that the state of stress, strain and temperature differs only slightly from the state in a known finite deformation. We suppose that the displacement vector is given by
$$\mathbf{v}(\theta_1, \theta_2, \theta_3) + \epsilon \mathbf{w}(\theta_1, \theta_2, \theta_3), \tag{3·1}$$
where $\mathbf{v}$ is the displacement vector of the known large deformation and ϵ is a constant which is small enough for squares and higher powers to be neglected. The analysis of strain is identical with that given in chapter IV of Green & Zerna (1954) (see also Green, Rivlin & Shield (1952)) and the relevant results will be quoted here. Much of the analysis of stress is also similar but now we must allow for temperature variations and must discuss the heat-conduction vector. We suppose that the temperature has the form
$$T(\theta_1, \theta_2, \theta_3) + \epsilon T'(\theta_1, \theta_2, \theta_3), \tag{3·2}$$
where T is the temperature distribution corresponding to the displacement vector $\mathbf{v}$. We denote base vectors, metric tensors, strain tensors and invariants, stress tensors corresponding

to (3·1) and (3·2) by $\mathbf{G}_i + \epsilon\mathbf{G}'_i$, $\mathbf{G}^i + \epsilon\mathbf{G}'^i$, $G_{ij} + \epsilon G'_{ij}$, $G^{ij} + \epsilon G'^{ij}$, $\gamma_{ij} + \epsilon\gamma'_{ij}$, $\gamma^i_j + \epsilon\gamma'^i_j$, $I_i + \epsilon I'_i$, and $\tau^{ij} + \epsilon\tau'^{ij}$, respectively. Hence

$$\left.\begin{aligned} \mathbf{G}'_i &= \mathbf{w}_{,i} = w_m\|_i\, \mathbf{G}^m = w^m\|_i\, \mathbf{G}_m, \\ \mathbf{w} &= w_m \mathbf{G}^m = w^m \mathbf{G}_m, \\ G'_{ij} &= \mathbf{G}_i . \mathbf{G}'_j + \mathbf{G}_j . \mathbf{G}'_i = w_i\|_j + w_j\|_i, \\ G' &= G G^{ij} G'_{ij}, \\ \mathbf{G}'^i &= G^{ij}\mathbf{G}'_j + G'^{ij}\mathbf{G}_j, \\ G'^{ij} &= -G^{ir}G^{js}G'_{rs}. \end{aligned}\right\} \tag{3·3}$$

Also

$$\left.\begin{aligned} I'_1 &= g^{rs}G'_{rs}, \\ I'_2 &= g_{rs}(G'^{rs}I_3 + G^{rs}I'_3), \\ I'_3 &= G'/g = I_3 G^{ij}G'_{ij}, \end{aligned}\right\} \tag{3·4}$$

and

$$\gamma'_{ij} = \tfrac{1}{2}G'_{ij}, \quad \gamma'^i_j = \tfrac{1}{2}g^{ir}G'_{rj}. \tag{3·5}$$

Formulae for τ'^{ij} are given in Green & Zerna, when the temperature is constant, for a body which is initially isotropic and either compressible or incompressible. When the temperature takes the form (3·2) extra terms are required in τ'^{ij} but since the derivation is straightforward we quote the final results without detailed working. Thus, for a body which is compressible and initially isotropic,

$$\tau'^{ij} = g^{ij}\Phi' + B^{ij}\Psi' + B'^{ij}\Psi + G'^{ij}p + G^{ij}p', \tag{3·6}$$

where

$$B'^{ij} = (g^{ij}g^{rs} - g^{ir}g^{js})\, G'_{rs} = \mathrm{e}^{irm}\,\mathrm{e}^{jsn}\, g_{rs}\, G'_{mn}/g, \tag{3·7}$$

and

$$\left.\begin{aligned} \Phi' &= AI'_1 + FI'_2 + EI'_3 - \frac{\Phi}{2I_3}I'_3 + LT', \\ \Psi' &= FI'_1 + BI'_2 + DI'_3 - \frac{\Psi}{2I_3}I'_3 + MT', \\ p' &= I_3(EI'_1 + DI'_2 + CI'_3) + \frac{p}{2I_3}I'_3 + NI_3T'. \end{aligned}\right\} \tag{3·8}$$

In (3·8)

$$\left.\begin{aligned} A &= \frac{2}{\sqrt{I_3}}\frac{\partial^2 W}{\partial I_1^2}, & B &= \frac{2}{\sqrt{I_3}}\frac{\partial^2 W}{\partial I_2^2}, & C &= \frac{2}{\sqrt{I_3}}\frac{\partial^2 W}{\partial I_3^2}, \\ D &= \frac{2}{\sqrt{I_3}}\frac{\partial^2 W}{\partial I_2 \partial I_3}, & E &= \frac{2}{\sqrt{I_3}}\frac{\partial^2 W}{\partial I_3 \partial I_1}, & F &= \frac{2}{\sqrt{I_3}}\frac{\partial^2 W}{\partial I_1 \partial I_2}, \\ L &= \frac{2}{\sqrt{I_3}}\frac{\partial^2 W}{\partial T \partial I_1}, & M &= \frac{2}{\sqrt{I_3}}\frac{\partial^2 W}{\partial T \partial I_2}, & N &= \frac{2}{\sqrt{I_3}}\frac{\partial^2 W}{\partial T \partial I_3}, \end{aligned}\right\} \tag{3·9}$$

the derivatives in (3·9) being evaluated at $\epsilon = 0$.

When the body is incompressible the formulae (3·6) still hold but now

$$\left.\begin{aligned} \Phi' &= AI'_1 + FI'_2 + LT', \\ \Psi' &= FI'_1 + BI'_2 + MT', \end{aligned}\right\} \tag{3·10}$$

where A, F, B are given by (3·9) with $I_3 = 1$, and p' is not given by (3·8) but is an arbitrary scalar function.

The stress equations of equilibrium are the same as those in Green & Zerna (1954) but here we obtain equations in a slightly different form. Corresponding to (3·1) and (3·2) $\mathbf{T}_i$ and ρ in (2·12) become $\mathbf{T}_i+\epsilon\mathbf{T}'_i$, and $\rho+\epsilon\rho'$, respectively. From (2·14) and (3·3) we see that

$$\mathbf{T}'_i = \surd(G)\,[\tau'^{im}+\tau^{ij}w^m\|_j+\tau^{im}w^j\|_j]\,\mathbf{G}_m. \tag{3·11}$$

Also, from (2·12) and (2·15) we have

$$\mathbf{T}'_{i,i}=0, \quad \rho'=-\rho w^i\|_i. \tag{3·12}$$

It follows from (3·11) and (3·12) that

$$\tau'^{ij}\|_i+[\tau^{im}w^j\|_m+\tau^{ij}w^m\|_m]\|_i=0. \tag{3·13}$$

Alternatively, using (2·13) we can reduce (3·13) to

$$\tau'^{ij}\|_i+\tau^{im}w^j\|_{mi}+\tau^{ij}w^m\|_{mi}=0. \tag{3·14}$$

Equation (3·14) is a more convenient form for equation (4·1·17) in chapter IV of Green & Zerna (1954).

Finally, we require expressions for the components of the heat-conduction vector which are denoted by $Q^i+\epsilon Q'^i$. A general expression for Q'^i can be found from (2·18) but in order to avoid undue complication in the analysis we restrict attention now to problems such that the deformation which corresponds to the displacement vector $\mathbf{v}$ (i.e. $\epsilon=0$) is one of equilibrium at *constant* temperature T. It follows that

$$Q^i=0, \quad -Q'^i=(\mathscr{C}_1\delta^i_j+\mathscr{C}_2\gamma^i_j+\mathscr{C}_3\gamma^i_m\gamma^m_j)\,T'\|^j, \tag{3·15}$$

where $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$ are now polynomials in I_1, I_2, I_3 with coefficients which are constants depending on the constant temperature T. In (3·15)

$$T'\|^j=G^{jr}T'\|_r=G^{jr}\frac{\partial T'}{\partial\theta^r}. \tag{3·16}$$

Since Q^i is zero, equations (2·17) and (2·21) yield

$$Q'^i\|_i=0 \quad \text{or} \quad \frac{\partial}{\partial\theta^i}(Q'^i\surd G)=0, \tag{3·17}$$

$$-Q'^iT'\|_i\geqslant 0. \tag{3·18}$$

4. Small deformation superposed on large uniform extensions

We now assume that the initial deformation is one of uniform extensions parallel to a set of rectangular Cartesian axes. Referred to this system of axes points of the body, after the initial deformation which corresponds to the displacement vector $\mathbf{v}$, are denoted by (x, y, z) and we choose θ_i so that

$$\theta_1=x, \quad \theta_2=y, \quad \theta_3=z. \tag{4·1}$$

Hence

$$G_{ij}=G^{ij}=\delta_{ij}, \quad G=1, \quad \mathbf{G}_m=\mathbf{G}^m, \tag{4·2}$$

and if the uniform extension ratios parallel to the axes are λ_1, λ_2, λ_3 then

$$g_{ij}=\begin{bmatrix}\dfrac{1}{\lambda_1^2}, & 0, & 0,\\ 0, & \dfrac{1}{\lambda_2^2}, & 0,\\ 0, & 0, & \dfrac{1}{\lambda_3^2},\end{bmatrix}, \quad g^{ij}=\begin{bmatrix}\lambda_1^2, & 0, & 0,\\ 0, & \lambda_2^2, & 0,\\ 0, & 0, & \lambda_3^2,\end{bmatrix}. \tag{4·3}$$

With these values of the metric tensors the special values assumed by the formulae of § 3 can be obtained without difficulty and results corresponding to $T' \equiv 0$ have already been given in chapter IV of Green & Zerna (1954). Consequently only the final formulae will be recorded here.

When the body is compressible

$$\left.\begin{aligned} \tau^{11} &= \Phi\lambda_1^2 + \Psi\lambda_1^2(\lambda_2^2+\lambda_3^2)+p, \\ \tau^{22} &= \Phi\lambda_2^2 + \Psi\lambda_2^2(\lambda_3^2+\lambda_1^2)+p, \\ \tau^{33} &= \Phi\lambda_3^2 + \Psi\lambda_3^2(\lambda_1^2+\lambda_2^2)+p, \\ \tau^{12} &= \tau^{23} = \tau^{31} = 0. \end{aligned}\right\} \tag{4·4}$$

$$\left.\begin{aligned} \tau'^{11} &= c_{11}\frac{\partial u}{\partial x}+c_{12}\frac{\partial v}{\partial y}+c_{13}\frac{\partial w}{\partial z}+\omega_1 T', \\ \tau'^{22} &= c_{21}\frac{\partial u}{\partial x}+c_{22}\frac{\partial v}{\partial y}+c_{23}\frac{\partial w}{\partial z}+\omega_2 T', \\ \tau'^{33} &= c_{31}\frac{\partial u}{\partial x}+c_{32}\frac{\partial v}{\partial y}+c_{33}\frac{\partial w}{\partial z}+\omega_3 T', \end{aligned}\right\} \tag{4·5}$$

$$\left.\begin{aligned} \tau'^{23} &= c_{44}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right), \\ \tau'^{31} &= c_{55}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right), \\ \tau'^{12} &= c_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right), \end{aligned}\right\} \tag{4·6}$$

where $\quad w^1 = w_1 = u, \quad w^2 = w_2 = v, \quad w^3 = w_3 = w,$

$$\begin{aligned} c_{11} &= -\tau^{11}+2A\lambda_1^4+2B\lambda_1^4(\lambda_2^2+\lambda_3^2)^2+2C\lambda_1^4\lambda_2^4\lambda_3^4 \\ &\quad +4D\lambda_1^4\lambda_2^2\lambda_3^2(\lambda_2^2+\lambda_3^2)+4E\lambda_1^4\lambda_2^2\lambda_3^2+4F\lambda_1^4(\lambda_2^2+\lambda_3^2), \\ c_{12} &= -\Phi\lambda_1^2+\Psi\lambda_1^2(\lambda_2^2-\lambda_3^2)+p+2A\lambda_1^2\lambda_2^2 \\ &\quad +2B\lambda_1^2\lambda_2^2(\lambda_2^2+\lambda_3^2)(\lambda_3^2+\lambda_1^2)+2C\lambda_1^4\lambda_2^4\lambda_3^4 \\ &\quad +2D\lambda_1^2\lambda_2^2\lambda_3^2(\lambda_2^2\lambda_3^2+\lambda_3^2\lambda_1^2+2\lambda_1^2\lambda_2^2) \\ &\quad +2E\lambda_1^2\lambda_2^2\lambda_3^2(\lambda_1^2+\lambda_2^2)+2F\lambda_1^2\lambda_2^2(\lambda_1^2+\lambda_2^2+2\lambda_3^2), \end{aligned} \tag{4·7}$$

and

$$\left.\begin{aligned} \omega_1 &= L\lambda_1^2+M\lambda_1^2(\lambda_2^2+\lambda_3^2)+N\lambda_1^2\lambda_2^2\lambda_3^2, \\ \omega_2 &= L\lambda_2^2+M\lambda_2^2(\lambda_3^2+\lambda_1^2)+N\lambda_1^2\lambda_2^2\lambda_3^2, \\ \omega_3 &= L\lambda_3^2+M\lambda_3^2(\lambda_1^2+\lambda_2^2)+N\lambda_1^2\lambda_2^2\lambda_3^2. \end{aligned}\right\} \tag{4·8}$$

Also c_{22}, c_{33} are obtained from c_{11} by cyclic permutation of λ_1, λ_2, λ_3 and τ^{11}, τ^{22}, τ^{33}, and c_{23}, c_{31} are obtained from c_{12} by cyclic permutation of λ_1, λ_2, λ_3. In addition

$$\left.\begin{aligned} c_{21}-c_{12} &= \tau^{11}-\tau^{22}, \\ c_{32}-c_{23} &= \tau^{22}-\tau^{33}, \\ c_{13}-c_{31} &= \tau^{33}-\tau^{11}, \end{aligned}\right\} \tag{4·9}$$

$$\left.\begin{aligned} c_{44} &= -\Psi\lambda_2^2\lambda_3^2-p, \\ c_{55} &= -\Psi\lambda_3^2\lambda_1^2-p, \\ c_{66} &= -\Psi\lambda_1^2\lambda_2^2-p. \end{aligned}\right\} \tag{4·10}$$

When the body is incompressible

$$\lambda_1\lambda_2\lambda_3 = 1, \quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0, \tag{4·11}$$

and then

$$\left.\begin{aligned} \tau'^{11} &= p'+a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial v}{\partial y}+a_{13}\frac{\partial w}{\partial z}+\omega_1 T', \\ \tau'^{22} &= p'+a_{12}\frac{\partial u}{\partial x}+a_{22}\frac{\partial v}{\partial y}+a_{23}\frac{\partial w}{\partial z}+\omega_2 T', \\ \tau'^{33} &= p'+a_{13}\frac{\partial u}{\partial x}+a_{23}\frac{\partial v}{\partial y}+a_{33}\frac{\partial w}{\partial z}+\omega_3 T', \end{aligned}\right\} \tag{4·12}$$

where

$$\left.\begin{aligned} a_{11} &= -2p+2\lambda_1^4\{A+B(\lambda_2^2+\lambda_3^2)^2+2F(\lambda_2^2+\lambda_3^2)\}, \\ a_{12} &= 2\lambda_1^2\lambda_2^2\{\Psi+A+B(\lambda_2^2+\lambda_3^2)(\lambda_3^2+\lambda_1^2)+F(\lambda_1^2+\lambda_2^2+2\lambda_3^2)\}, \end{aligned}\right\} \tag{4·13}$$

a_{22}, a_{33} and a_{23}, a_{13} being obtained from a_{11}, a_{12} respectively by cyclic permutation of λ_1, λ_2, λ_3. The remaining stress components are given by (4·6) and (4·10) and ω_1 reduces to

$$\omega_1 = L\lambda_1^2+M\lambda_1^2(\lambda_2^2+\lambda_3^2), \tag{4·14}$$

with ω_2, ω_3 obtained from this by cyclic permutation of λ_1, λ_2, λ_3.

We observe that all the coefficients c_{ij}, a_{ij}, ω_i are constants and p' in (4·12) is an arbitrary scalar function.

For some purposes it is more convenient to have stress components referred to rectangular Cartesian co-ordinates (x, y, z) and these can be obtained as in Green & Zerna. Denoting the stress components by $t^{ij}+\epsilon t'^{ij}$ we have

$$t^{ij} = \tau^{ij}, \quad t'^{rs} = \tau'^{rs}+\tau^{ms}\frac{\partial w_r}{\partial \theta^m}+\tau^{rm}\frac{\partial w_s}{\partial \theta^m}, \tag{4·15}$$

and (3·14) are replaced by

$$\frac{\partial t'^{ij}}{\partial \theta^j} = 0. \tag{4·16}$$

In view of (4·4) equations (4·15) give

$$\left.\begin{aligned} t'^{11} &= \tau'^{11}+2\tau^{11}\frac{\partial u}{\partial x}, \\ t'^{22} &= \tau'^{22}+2\tau^{22}\frac{\partial v}{\partial y}, \\ t'^{33} &= \tau'^{33}+2\tau^{33}\frac{\partial w}{\partial z}, \\ t'^{12} &= \tau'^{12}+\tau^{11}\frac{\partial v}{\partial x}+\tau^{22}\frac{\partial u}{\partial y}, \\ t'^{23} &= \tau'^{23}+\tau^{22}\frac{\partial w}{\partial y}+\tau^{33}\frac{\partial v}{\partial z}, \\ t'^{31} &= \tau'^{31}+\tau^{33}\frac{\partial u}{\partial z}+\tau^{11}\frac{\partial w}{\partial x}. \end{aligned}\right\} \tag{4·17}$$

To complete the formulae we see from (3·15) that

$$Q'^1 = -r_1\frac{\partial T'}{\partial x}, \quad Q'^2 = -r_2\frac{\partial T'}{\partial y}, \quad Q'^3 = -r_3\frac{\partial T'}{\partial z}, \tag{4·18}$$

where
$$r_i = \mathscr{C}_1 + \tfrac{1}{2}\mathscr{C}_2(\lambda_i^2-1) + \tfrac{1}{4}\mathscr{C}_3(\lambda_i^2-1)^2, \tag{4·19}$$

and i is not summed in (4·19), so that r_i are constants. Equations (3·17) and (4·18) now yield

$$r_1\frac{\partial^2 T'}{\partial x^2} + r_2\frac{\partial^2 T'}{\partial y^2} + r_3\frac{\partial^2 T'}{\partial z^2} = 0. \tag{4·20}$$

Finally, we conclude from (3·18) and (4·18) that

$$r_1 \geqslant 0, \quad r_2 \geqslant 0, \quad r_3 \geqslant 0. \tag{4·21}$$

5. Two equal extension ratios: compressible case

We consider now a special case of the preceding theory in which the extension ratios λ_1, λ_2 are equal so that

$$\lambda_1 = \lambda_2, \quad \omega_1 = \omega_2, \quad r_1 = r_2, \quad \tau^{11} = \tau^{22}. \tag{5·1}$$

The formulae (4·7), (4·9), (4·10) for the constants c_{ij} simplify but details are omitted.† Values of t'^{ij} can be found from (4·5), (4·6) and (4·15). Thus

$$\left.\begin{aligned} t'^{11} &= d_{11}\frac{\partial u}{\partial x} + d_{12}\frac{\partial v}{\partial y} + d_{13}\frac{\partial w}{\partial z} + \omega_1 T', \\ t'^{22} &= d_{12}\frac{\partial u}{\partial x} + d_{11}\frac{\partial v}{\partial y} + d_{13}\frac{\partial w}{\partial z} + \omega_1 T', \\ t'^{33} &= d_{31}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + d_{33}\frac{\partial w}{\partial z} + \omega_3 T', \end{aligned}\right\} \tag{5·2}$$

$$\left.\begin{aligned} t'^{12} &= \tfrac{1}{2}(d_{11}-d_{12})\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \\ t'^{23} &= d_{44}\frac{\partial v}{\partial z} + d_{55}\frac{\partial w}{\partial y}, \\ t'^{31} &= d_{44}\frac{\partial u}{\partial z} + d_{55}\frac{\partial w}{\partial x}, \end{aligned}\right\} \tag{5·3}$$

where‡
$$\left.\begin{aligned} d_{11} &= c_{11} + 2\tau^{11}, \quad d_{33} = c_{33} + 2\tau^{33}, \\ d_{12} &= c_{12}, \quad d_{13} = c_{13}, \quad d_{31} = c_{31}, \\ d_{11} - d_{12} &= c_{11} - c_{12} + 2\tau^{11} = 2\lambda_1^2(\Phi + \lambda_3^2\Psi), \\ d_{44} &= c_{44} + \tau^{33} = \lambda_3^2(\Phi + \lambda_1^2\Psi), \\ d_{55} &= c_{44} + \tau^{11} = \lambda_1^2(\Phi + \lambda_1^2\Psi), \\ d_{55} - d_{44} &= c_{31} - c_{13} = d_{31} - d_{13} = \tau^{11} - \tau^{33}. \end{aligned}\right\} \tag{5·4}$$

Also, the temperature equation (4·20), reduces to

$$r_1\nabla_1^2 T' + r_3\frac{\partial^2 T'}{\partial z^2} = 0, \quad \nabla_1^2 T' = \frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y^2}. \tag{5·5}$$

† The special values are given in Green & Zerna.
‡ See Green & Zerna for details.

We first solve the temperature problem (5·5) subject to suitable boundary conditions and then (5·2), (5·3) and (4·16) give three differential equations for u, v, w, namely

$$d_{11}\frac{\partial^2 u}{\partial x^2}+\tfrac{1}{2}(d_{11}-d_{12})\frac{\partial^2 u}{\partial y^2}+d_{44}\frac{\partial^2 u}{\partial z^2}+\tfrac{1}{2}(d_{11}+d_{12})\frac{\partial^2 v}{\partial x\,\partial y}+(d_{13}+d_{55})\frac{\partial^2 w}{\partial x\,\partial z}+\omega_1\frac{\partial T'}{\partial x}=0, \qquad (5·6)$$

$$\tfrac{1}{2}(d_{11}-d_{12})\frac{\partial^2 v}{\partial x^2}+d_{11}\frac{\partial^2 v}{\partial y^2}+d_{44}\frac{\partial^2 v}{\partial z^2}+\tfrac{1}{2}(d_{11}+d_{12})\frac{\partial^2 u}{\partial x\,\partial y}+(d_{13}+d_{55})\frac{\partial^2 w}{\partial y\,\partial z}+\omega_1\frac{\partial T'}{\partial y}=0, \qquad (5·7)$$

$$d_{55}\nabla_1^2 w+d_{33}\frac{\partial^2 w}{\partial z^2}+(d_{13}+d_{55})\left(\frac{\partial^2 u}{\partial x\,\partial z}+\frac{\partial^2 v}{\partial y\,\partial z}\right)+\omega_3\frac{\partial T'}{\partial z}=0. \qquad (5·8)$$

In deriving (5·8) we have used the last relation in (5·4).

Since T' is assumed to be known we first seek a particular solution of (5·6) to (5·8) which expresses u, v, w in terms of T'. The general solution is then found by adding to this particular solution the general solution of (5·6) to (5·8) when $T'=0$.

Suppose a particular solution can be expressed in the form

$$u=l\frac{\partial\chi}{\partial x},\quad v=l\frac{\partial\chi}{\partial y},\quad w=m\frac{\partial\chi}{\partial z}. \qquad (5·9)$$

Then (5·6) to (5·8) are satisfied provided that

$$\left.\begin{aligned} ld_{11}\nabla_1^2\chi+[ld_{44}+m(d_{13}+d_{55})]\frac{\partial^2\chi}{\partial z^2}+\omega_1 T'&=0,\\ [l(d_{13}+d_{55})+md_{55}]\nabla_1^2\chi+md_{33}\frac{\partial^2\chi}{\partial z^2}+\omega_3 T'&=0.\end{aligned}\right\} \qquad (5·10)$$

Recalling (5·5) we assume that χ is any particular integral of the equations†

$$r_1\nabla_1^2\chi+r_3\frac{\partial^2\chi}{\partial z^2}=0,\quad \frac{\partial^2\chi}{\partial z^2}=T'. \qquad (5·11)$$

Equations (5·10) and (5·11) are compatible if

$$\left.\begin{aligned} ld_{11}d_{55}(r_1\nu_1-r_3)(r_1\nu_2-r_3)&=\omega_3 r_1^2(d_{13}+d_{55})+\omega_1 r_1(r_3 d_{55}-r_1 d_{33}),\\ md_{11}d_{55}(r_1\nu_1-r_3)(r_1\nu_2-r_3)&=\omega_3 r_1(r_3 d_{11}-r_1 d_{44})-\omega_1 r_1 r_3(d_{13}+d_{55}),\end{aligned}\right\} \qquad (5·12)$$

where ν_1, ν_2 are roots of the quadratic equation (cf. Green & Zerna, p. 131)

$$d_{11}d_{55}\nu^2+\{(d_{13}+d_{55})^2-d_{11}d_{33}-d_{44}d_{55}\}\nu+d_{33}d_{44}=0. \qquad (5·13)$$

The above particular solution is degenerate‡ when $r_3/r_1=\nu_1$ or ν_2 but in this case there is a particular solution of the form

$$u=lz\frac{\partial^2\chi}{\partial x\,\partial z},\quad v=lz\frac{\partial^2\chi}{\partial y\,\partial z},\quad w=m_1 z\frac{\partial^2\chi}{\partial z^2}+m_2\frac{\partial\chi}{\partial z}, \qquad (5·14)$$

where χ is any particular integral of (5·11) and l, m_1, m_2 are given by

$$l=\frac{r_1[(r_1 d_{33}-r_3 d_{55})\,\omega_1-(d_{13}+d_{55})\,r_1\omega_3]}{2(r_3^2 d_{11}d_{55}-r_1^2 d_{33}d_{44})}, \qquad (5·15)$$

$$m_1=\frac{r_1[(d_{13}+d_{55})\,r_3\omega_1-(r_3 d_{11}-r_1 d_{44})\,\omega_3]}{2(r_3^2 d_{11}d_{55}-r_1^2 d_{33}d_{44})}, \qquad (5·16)$$

$$m_2=\frac{\omega_1(r_1 d_{44}-r_3 d_{11})(r_1 d_{33}+r_3 d_{55})+\omega_3(d_{13}+d_{55})\,r_1(r_1 d_{44}+r_3 d_{11})}{2(r_3^2 d_{11}d_{55}-r_1^2 d_{33}d_{44})(d_{13}+d_{55})}, \qquad (5·17)$$

† For the present we assume that it is possible to find χ from (5·11); this is certainly the case in the examples studied later in the paper.

‡ The solution, in this form, is also degenerate if $d_{11}=0$ or $d_{55}=0$ but we assume that these quantities are non-zero.

where, writing r_3/r_1 for ν in (5·13), we have

$$\frac{(d_{13}+d_{55})\,r_3}{r_1 d_{33}-r_3 d_{55}} = -\frac{r_1 d_{44}-r_3 d_{11}}{(d_{13}+d_{55})\,r_1} = r \quad \text{(say)}. \tag{5·18}$$

If, in addition, the quadratic (5·13) has equal roots we see that

$$r^2 = \frac{d_{44}}{d_{55}}, \quad \left(\frac{r_3}{r_1}\right)^2 = \frac{d_{33}d_{44}}{d_{11}d_{55}}, \tag{5·19}$$

and hence the particular solution (5·14) degenerates. In this case there is a particular solution of the form

$$\left.\begin{aligned} u &= lz^2\frac{\partial^3\chi}{\partial x\,\partial z^2},\\ v &= lz^2\frac{\partial^3\chi}{\partial y\,\partial z^2},\\ w &= m_1 z^2\frac{\partial^3\chi}{\partial z^3}+m_2 z\frac{\partial^2\chi}{\partial z^2}+m_3\frac{\partial\chi}{\partial z}, \end{aligned}\right\} \tag{5·20}$$

where χ is any particular integral of (5·11) and

$$\left.\begin{aligned} l &= \frac{(d_{13}+d_{55})\,(\omega_3 r-\omega_1 r_3/r_1)}{8d_{33}d_{44}r},\\ m_1 &= lr,\\ m_2 &= -\frac{(r_1 d_{44}+r_3 d_{11})\,(\omega_3 r-\omega_1 r_3/r_1)}{4d_{33}d_{44}rr_1},\\ m_3 &= -\frac{\omega_1}{d_{13}+d_{55}}+\frac{rr_1(\omega_3 r-\omega_1 r_3/r_1)}{4d_{44}r_3}. \end{aligned}\right\} \tag{5·21}$$

It now remains to find the general solution of (5·6) to (5·8) when $T' = 0$. Put

$$u = \frac{\partial\phi_1}{\partial x}+\frac{\partial\phi_3}{\partial y}, \quad v = \frac{\partial\phi_1}{\partial y}-\frac{\partial\phi_3}{\partial x}, \quad w = \frac{\partial\phi_2}{\partial z}. \tag{5·22}$$

If u, v, w are given, then

$$\phi_1 = \phi_1'+\frac{\partial\psi_1}{\partial y}, \quad \phi_3 = \phi_3'-\frac{\partial\psi_1}{\partial x}, \quad \phi_2 = \phi_2'+\psi_2, \tag{5·23}$$

where ϕ_1', ϕ_2', ϕ_3' are any particular integrals of the equations

$$\nabla_1^2\phi_1' = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}, \quad \nabla_1^2\phi_3' = \frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}, \quad \frac{\partial\phi_2'}{\partial z} = w. \tag{5·24}$$

Also ψ_1 is an arbitrary function of (x, y, z) subject to the condition

$$\nabla_1^2\psi_1 = 0, \tag{5·25}$$

and ψ_2 is an arbitrary function of (x, y). We shall not examine in detail here the conditions under which ϕ_1', ϕ_2', ϕ_3' can be found although this is not a difficult problem for the regions of space which we consider later in the paper. The functions ψ_1, ψ_2 do not contribute to the displacements and may therefore be omitted from ϕ_1, ϕ_2, ϕ_3.

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On substituting (5·22) in (5·6) to (5·8) we obtain the equations

$$\frac{\partial}{\partial x}\left[d_{11}\nabla_1^2\phi_1+d_{44}\frac{\partial^2\phi_1}{\partial z^2}+(d_{13}+d_{55})\frac{\partial^2\phi_2}{\partial z^2}\right]+\frac{\partial}{\partial y}\left[\tfrac{1}{2}(d_{11}-d_{12})\nabla_1^2\phi_3+d_{44}\frac{\partial^2\phi_3}{\partial z^2}\right]=0, \qquad (5\cdot26)$$

$$\frac{\partial}{\partial y}\left[d_{11}\nabla_1^2\phi_1+d_{44}\frac{\partial^2\phi_1}{\partial z^2}+(d_{13}+d_{55})\frac{\partial^2\phi_2}{\partial z^2}\right]-\frac{\partial}{\partial x}\left[\tfrac{1}{2}(d_{11}-d_{12})\nabla_1^2\phi_3+d_{44}\frac{\partial^2\phi_3}{\partial z^2}\right]=0, \qquad (5\cdot27)$$

$$\frac{\partial}{\partial z}\left[(d_{13}+d_{55})\nabla_1^2\phi_1+d_{55}\nabla_1^2\phi_2+d_{33}\frac{\partial^2\phi_2}{\partial z^2}\right]=0. \qquad (5\cdot28)$$

Hence

$$\left.\begin{aligned}\tfrac{1}{2}(d_{11}-d_{12})\nabla_1^2\phi_3+d_{44}\frac{\partial^2\phi_3}{\partial z^2}&=-d_{44}\frac{\partial f_1}{\partial x},\\ d_{11}\nabla_1^2\phi_1+d_{44}\frac{\partial^2\phi_1}{\partial z^2}+(d_{13}+d_{55})\frac{\partial^2\phi_2}{\partial z^2}&=d_{44}\frac{\partial f_1}{\partial y},\\ (d_{13}+d_{55})\nabla_1^2\phi_1+d_{55}\nabla_1^2\phi_2+d_{33}\frac{\partial^2\phi_2}{\partial z^2}&=d_{55}f_2(x,y),\end{aligned}\right\} \qquad (5\cdot29)$$

where f_2 is an arbitrary function of (x, y) and f_1 is an arbitrary function of (x, y, z) satisfying the equation

$$\nabla_1^2 f_1 = 0. \qquad (5\cdot30)$$

Let $g(x, y, z)$ be any particular integral of the equation†

$$\frac{\partial^2 g}{\partial z^2} = f_1(x, y, z). \qquad (5\cdot31)$$

Then

$$\frac{\partial^2}{\partial z^2}(\nabla_1^2 g) = \nabla_1^2\frac{\partial^2 g}{\partial z^2} = \nabla_1^2 f_1 = 0, \qquad (5\cdot32)$$

and hence

$$\nabla_1^2 g = \alpha(x, y)\, z + \beta(x, y), \qquad (5\cdot33)$$

where α, β are arbitrary functions of (x, y). Next define

$$g_1(x, y, z) = g(x, y, z) + A(x, y)\, z + B(x, y), \qquad (5\cdot34)$$

where

$$\nabla_1^2 A = -\alpha, \quad \nabla_1^2 B = -\beta. \qquad (5\cdot35)$$

It follows that

$$\frac{\partial^2 g_1}{\partial z^2} = \frac{\partial^2 g}{\partial z^2} = f_1, \qquad (5\cdot36)$$

and

$$\nabla_1^2 g_1 = 0. \qquad (5\cdot37)$$

We can now take a particular integral of the equations (5·29) in the form

$$\phi_1 = \frac{\partial g_1}{\partial y}, \quad \phi_3 = -\frac{\partial g_1}{\partial x}, \quad \phi_2 = g_2(x, y), \qquad (5\cdot38)$$

where g_1 satisfies (5·37) and g_2 is a particular integral of the equation

$$\nabla_1^2 g_2 = f_2(x, y). \qquad (5\cdot39)$$

The particular integrals (5·38), however, do not contribute to the displacements and may be ignored so that, without loss of generality, we may put $f_1 = f_2 \equiv 0$ in the right-hand side of (5·29).

† Again, this can be found in the special problems considered later.

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In order to complete the solution we observe that equation (5·13) may be written as

$$\frac{d_{11}\nu - d_{44}}{d_{13}+d_{55}} = \frac{(d_{13}+d_{55})\,\nu}{d_{33}-d_{55}\nu} = k, \tag{5·40}$$

so that corresponding to distinct roots ν_1, ν_2 of (5·13) we have distinct values k_1, k_2 of k. Alternatively (5·40) may be put in the form

$$\frac{d_{44}+(d_{13}+d_{55})\,k}{d_{11}} = \frac{d_{33}k}{d_{55}k+d_{13}+d_{55}} = \nu. \tag{5·41}$$

Without loss of generality we may now put

$$\phi_1 = \chi_1+\chi_2, \quad \phi_2 = k_1\chi_1+k_2\chi_2, \tag{5·42}$$

and substitute into the second and third of equations (5·29). Thus, using (5·41), we find that

$$\left.\begin{aligned}\left(\nabla_1^2+\nu_1\frac{\partial^2}{\partial z^2}\right)\chi_1+\left(\nabla_1^2+\nu_2\frac{\partial^2}{\partial z^2}\right)\chi_2 = 0,\\ \frac{k_1}{\nu_1}\left(\nabla_1^2+\nu_1\frac{\partial^2}{\partial z^2}\right)\chi_1+\frac{k_2}{\nu_2}\left(\nabla_1^2+\nu_2\frac{\partial^2}{\partial z^2}\right)\chi_2 = 0.\end{aligned}\right\} \tag{5·43}$$

If $k_1/\nu_1 \neq k_2/\nu_2$ it follows that

$$\left(\nabla_1^2+\nu_\alpha\frac{\partial^2}{\partial z^2}\right)\chi_\alpha = 0 \quad (\alpha = 1, 2). \tag{5·44}$$

To complete the notation we put

$$\phi_3 = \chi_3, \tag{5·45}$$

where, from (5·29),

$$\left(\nabla_1^2+\nu_3\frac{\partial^2}{\partial z^2}\right)\chi_3 = 0, \quad \nu_3 = \frac{2d_{44}}{d_{11}-d_{12}}. \tag{5·46}$$

Thus the general solution† in the case $T' = 0$ to the equations of equilibrium (5·6), (5·7) and (5·8) is given by

$$\left.\begin{aligned}u &= \frac{\partial\chi_1}{\partial x}+\frac{\partial\chi_2}{\partial x}+\frac{\partial\chi_3}{\partial y},\\ v &= \frac{\partial\chi_1}{\partial y}+\frac{\partial\chi_2}{\partial y}-\frac{\partial\chi_3}{\partial x},\\ w &= k_1\frac{\partial\chi_1}{\partial z}+k_2\frac{\partial\chi_2}{\partial z},\end{aligned}\right\} \tag{5·47}$$

where χ_1, χ_2, and χ_3 satisfy (5·44) and (5·46) respectively and k_1, k_2 are defined by (5·40).

When the quadratic (5·13) (or (5·40)) has equal roots, k_1 and k_2 are equal and the solution (5·42) becomes incomplete. In this case

$$\nu_1^2 = \nu_2^2 = \frac{d_{33}d_{44}}{d_{11}d_{55}}, \quad k_1^2 = k_2^2 = \frac{d_{44}}{d_{55}}, \tag{5·48}$$

$$d_{13}+d_{55} = \frac{d_{11}\nu_1-d_{44}}{k_1} = \frac{(d_{33}-d_{55}\nu_1)\,k_1}{\nu_1}, \tag{5·49}$$

† The solution is degenerate in certain circumstances, one such case being considered below. In most problems the solution in any degenerate case can be found by a limiting process from the general solution. The solution (5·47) is more complete than that given by Green & Zerna (1954).

and the last two equations in (5·29) become

$$\left.\begin{aligned} d_{11}\left(\nabla_1^2\phi_1+\frac{\nu_1}{k_1}\frac{\partial^2\phi_2}{\partial z^2}\right)+\frac{d_{44}}{k_1}\frac{\partial^2}{\partial z^2}(k_1\phi_1-\phi_2) &= 0,\\ \frac{d_{33}k_1}{\nu_1}\left(\nabla_1^2\phi_1+\frac{\nu_1}{k_1}\frac{\partial^2\phi_2}{\partial z^2}\right)-d_{55}\nabla_1^2(k_1\phi_1-\phi_2) &= 0.\end{aligned}\right\} \tag{5·50}$$

Hence, using (5·48),

$$\left(\nabla_1^2+\nu_1\frac{\partial^2}{\partial z^2}\right)(k_1\phi_1-\phi_2) = 0, \tag{5·51}$$

and therefore

$$k_1\phi_1-\phi_2 = \frac{2d_{11}\nu_1}{d_{13}+d_{55}}\chi_1, \tag{5·52}$$

where

$$\left(\nabla_1^2+\nu_1\frac{\partial^2}{\partial z^2}\right)\chi_1 = 0. \tag{5·53}$$

Then, substituting from (5·52) into (5·50), we see that

$$\left(\nabla_1^2+\nu_1\frac{\partial^2}{\partial z^2}\right)\phi_1 = 2\nu_1\frac{\partial^2\chi_1}{\partial z^2}, \tag{5·54}$$

and the complete integral is

$$\phi_1 = z\frac{\partial\chi_1}{\partial z}+\chi_2, \tag{5·55}$$

where

$$\left(\nabla_1^2+\nu_1\frac{\partial^2}{\partial z^2}\right)\chi_2 = 0. \tag{5·56}$$

Also, from (5·52) and (5·55),

$$\phi_2 = k_1 z\frac{\partial\chi_1}{\partial z}+k_1\chi_2-\frac{2d_{11}\nu_1}{d_{13}+d_{55}}\chi_1. \tag{5·57}$$

Collecting all the contributions the complete solution of the homogeneous problem when $\nu_1 = \nu_2$ is

$$\left.\begin{aligned} u &= z\frac{\partial^2\chi_1}{\partial x\,\partial z}+\frac{\partial\chi_2}{\partial x}+\frac{\partial\chi_3}{\partial y},\\ v &= z\frac{\partial^2\chi_1}{\partial y\,\partial z}+\frac{\partial\chi_2}{\partial y}-\frac{\partial\chi_3}{\partial x},\\ w &= k_1 z\frac{\partial^2\chi_1}{\partial z^2}-k_1\left(\frac{d_{11}\nu_1+d_{44}}{d_{11}\nu_1-d_{44}}\right)\frac{\partial\chi_1}{\partial z}+k_1\frac{\partial\chi_2}{\partial z},\end{aligned}\right\} \tag{5·58}$$

where k_1, ν_1 are given by (5·48) and χ_1, χ_2 satisfy the equations (5·53) and (5·56). Also χ_3 satisfies (5·46).

Thus the general solution to the equations of equilibrium (5·6), (5·7) and (5·8) is given by addition of the particular solution (5·9) or (5·14) or (5·20) to the general solution (5·47) or (5·58).

For completeness, assuming that $r_3/r_1 \neq \nu_1$ or ν_2 and $\nu_1 \neq \nu_2$, the general solution is

$$\left.\begin{aligned} u &= \frac{\partial\chi_1}{\partial x}+\frac{\partial\chi_2}{\partial x}+\frac{\partial\chi_3}{\partial y}+l\frac{\partial\chi}{\partial x},\\ v &= \frac{\partial\chi_1}{\partial y}+\frac{\partial\chi_2}{\partial y}-\frac{\partial\chi_3}{\partial x}+l\frac{\partial\chi}{\partial y},\\ w &= k_1\frac{\partial\chi_1}{\partial z}+k_2\frac{\partial\chi_2}{\partial z}+m\frac{\partial\chi}{\partial z},\end{aligned}\right\} \tag{5·59}$$

where the potential functions χ_i satisfy equations (5·44) and (5·46), χ satisfies (5·11), and k_1, k_2, l, m are given by (5·40) and (5·12) respectively.

6. Two equal extension ratios: incompressible case

We consider a special case of the theory in § 4 for an incompressible body in which the extension ratios λ_1, λ_2 are equal so that (5·1) holds and

$$\lambda_3 \lambda_1^2 = 1. \tag{6·1}$$

Then, as in Green & Zerna (1954, p. 132) the values of t'^{ij} can be found from (4·11), (4·12) and (4·17), so that

$$\left.\begin{aligned} t'^{11} &= p' + a\frac{\partial u}{\partial x} + b\frac{\partial v}{\partial y} + \omega_1 T',\\ t'^{22} &= p' + b\frac{\partial u}{\partial x} + a\frac{\partial v}{\partial y} + \omega_1 T',\\ t'^{33} &= p' + c\frac{\partial w}{\partial z} + \omega_3 T',\end{aligned}\right\} \tag{6·2}$$

where

$$\left.\begin{aligned} a &= 2\tau^{11} - 2\Psi\lambda_1^2\lambda_3^2 - 2p + 2\lambda_1^2(\lambda_1^2-\lambda_3^2)\{A + B\lambda_1^2(\lambda_1^2+\lambda_3^2) + F(2\lambda_1^2+\lambda_3^2)\},\\ b &= 2\lambda_1^2(\lambda_1^2-\lambda_3^2)\{\Psi + A + B\lambda_1^2(\lambda_1^2+\lambda_3^2) + F(2\lambda_1^2+\lambda_3^2)\},\\ c &= 2\tau^{33} - 2\Psi\lambda_1^2\lambda_3^2 - 2p + 2\lambda_3^2(\lambda_3^2-\lambda_1^2)(A + 2B\lambda_1^4 + 3F\lambda_1^2),\\ \omega_1 &= L\lambda_1^2 + M\lambda_1^2(\lambda_1^2+\lambda_3^2), \quad \omega_3 = L\lambda_3^2 + 2M\lambda_3^2\lambda_1^2.\end{aligned}\right\} \tag{6·3}$$

The components t'^{12}, t'^{23}, t'^{31} are given by (5·3) where now

$$d_{11} - d_{12} = a - b, \quad d_{44} = \lambda_3^2(\Phi + \lambda_1^2\Psi), \quad d_{55} = \lambda_1^2(\Phi + \lambda_1^2\Psi). \tag{6·4}$$

Since the deformed body is assumed to be in equilibrium under a steady-state distribution of temperature, equation (5·5) is solved subject to given boundary conditions and then the incompressibility condition (4·11) and equations (4·16), (5·3) and (6·2) yield four differential equations for u, v, w, p'. These may be written in the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{6·5}$$

$$\left.\begin{aligned} &\frac{\partial}{\partial x}(p' + \omega_1 T') + (a - d_{55})\frac{\partial^2 u}{\partial x^2} + \tfrac{1}{2}(a-b)\frac{\partial^2 u}{\partial y^2} + d_{44}\frac{\partial^2 u}{\partial z^2} + \{\tfrac{1}{2}(a+b) - d_{55}\}\frac{\partial^2 v}{\partial x\,\partial y} = 0,\\ &\frac{\partial}{\partial y}(p' + \omega_1 T') + \{\tfrac{1}{2}(a+b) - d_{55}\}\frac{\partial^2 u}{\partial x\,\partial y} + \tfrac{1}{2}(a-b)\frac{\partial^2 v}{\partial x^2} + (a - d_{55})\frac{\partial^2 v}{\partial y^2} + d_{44}\frac{\partial^2 v}{\partial z^2} = 0,\\ &\frac{\partial}{\partial z}(p' + \omega_3 T') + d_{55}\nabla_1^2 w + (c - d_{44})\frac{\partial^2 w}{\partial z^2} = 0.\end{aligned}\right\} \tag{6·6}$$

As in the compressible case we seek a particular integral of (6·5) and (6·6) expressing u, v, w, p' in terms of T'. The general solution is then obtained by combining this with a general solution to the homogeneous problem given by $T' = 0$.

There exists a particular solution of (6·5) and (6·6) of the form

$$\left.\begin{aligned} &u = l\frac{\partial \chi}{\partial x}, \quad v = l\frac{\partial \chi}{\partial y}, \quad w = m\frac{\partial \chi}{\partial z},\\ &p' = -\frac{r_3\omega_1[r_1(d_{44}-c) + r_3 d_{55}] + r_1\omega_3[r_1 d_{44} + r_3(d_{55}-a)]}{d_{55}(r_1 k_1 - r_3)(r_1 k_2 - r_3)}\frac{\partial^2 \chi}{\partial z^2},\end{aligned}\right\} \tag{6·7}$$

where, in view of (5·5),

$$r_1 \nabla_1^2 \chi + r_3 \frac{\partial^2 \chi}{\partial z^2} = 0, \quad \frac{\partial^2 \chi}{\partial z^2} = T', \tag{6·8}$$

and

$$m = \frac{r_3}{r_1} l = \frac{r_1 r_3(\omega_3 - \omega_1)}{d_{55}(r_1 k_1 - r_3)(r_1 k_2 - r_3)}. \tag{6·9}$$

In (6·7) to (6·9), k_1, k_2 are the roots, assumed at present to be distinct, of the quadratic equation

$$k^2 d_{55} + k(d_{44} + d_{55} - a - c) + d_{44} = 0. \tag{6·10}$$

In order to find a general solution to equations (6·5) and (6·6) we again make the substitution (5·22) when $T' = 0$, leaving aside any discussion about the possible limitations thereby imposed on the region of space considered, since in the special applications made in this paper the form of solution is satisfactory. By analysis similar to that given in § 5 we may show that the solution (5·22) is suitable provided

$$\left.\begin{aligned} \nabla_1^2 \phi_1 + \frac{\partial^2 \phi_2}{\partial z^2} &= 0, \\ p' + (a - d_{55}) \nabla_1^2 \phi_1 + d_{44} \frac{\partial^2 \phi_1}{\partial z^2} &= 0, \\ p' + d_{55} \nabla_1^2 \phi_2 + (c - d_{44}) \frac{\partial^2 \phi_2}{\partial z^2} &= 0, \end{aligned}\right\} \tag{6·11}$$

$$\tfrac{1}{2}(a-b) \nabla_1^2 \phi_3 + d_{44} \frac{\partial^2 \phi_3}{\partial z^2} = 0. \tag{6·12}$$

Without loss of generality we now put

$$\phi_1 = \chi_1 + \chi_2, \quad \phi_2 = k_1 \chi_1 + k_2 \chi_2, \quad \phi_3 = \chi_3, \tag{6·13}$$

where k_1, k_2 are the (distinct) roots of (6·10). It follows from (6·11) and (6·12) that

$$\left(\nabla_1^2 + k_\alpha \frac{\partial^2}{\partial z^2}\right) \chi_\alpha = 0 \quad (\alpha = 1, 2, 3), \tag{6·14}$$

where

$$k_3 = \frac{2d_{44}}{a-b}, \tag{6·15}$$

and

$$\left.\begin{aligned} p' &= k_1(k_1 d_{55} + d_{44} - c) \frac{\partial^2 \chi_1}{\partial z^2} + k_2(k_2 d_{55} + d_{44} - c) \frac{\partial^2 \chi_2}{\partial z^2}, \\ &= (k_1 a - k_1 d_{55} - d_{44}) \frac{\partial^2 \chi_1}{\partial z^2} + (k_2 a - k_2 d_{55} - d_{44}) \frac{\partial^2 \chi_2}{\partial z^2}. \end{aligned}\right\} \tag{6·16}$$

Thus the solution in the homogeneous case is given by†

$$\left.\begin{aligned} u &= \frac{\partial \chi_1}{\partial x} + \frac{\partial \chi_2}{\partial x} + \frac{\partial \chi_3}{\partial y}, \\ v &= \frac{\partial \chi_1}{\partial y} + \frac{\partial \chi_2}{\partial y} - \frac{\partial \chi_3}{\partial x}, \\ w &= k_1 \frac{\partial \chi_1}{\partial z} + k_2 \frac{\partial \chi_2}{\partial z}, \\ p' &= (d_{44} - c + d_{55} k_1) k_1 \frac{\partial^2 \chi_1}{\partial z^2} + (d_{44} - c + d_{55} k_2) k_2 \frac{\partial^2 \chi_2}{\partial z^2}, \end{aligned}\right\} \tag{6·17}$$

where the potential functions χ_i satisfy (6·14).

† This solution is more complete than that given by Green & Zerna (1954).

As already indicated the above solutions fail in certain special cases, the most important being when r_3/r_1 is equal to k_1 or k_2, or when the roots k_1, k_2 of the quadratic (6·10) are equal.†

In the first case when r_3/r_1 is equal to k_1 or k_2, we find there exists a particular solution of the form

$$\left.\begin{aligned} u &= lz\frac{\partial^2\chi}{\partial x\,\partial z}, \quad v = lz\frac{\partial^2\chi}{\partial y\,\partial z}, \quad w = mz\frac{\partial^2\chi}{\partial z^2} - m\frac{\partial\chi}{\partial z}, \\ p' &= \frac{r_1(\omega_1-\omega_3)\{d_{44}r_1+(d_{55}-a)\,r_3\}}{2(d_{44}r_1^2-d_{55}r_3^2)}\,z\frac{\partial^3\chi}{\partial z^3} + \frac{\omega_1 d_{55}r_3^2-\omega_3 d_{44}r_1^2}{d_{44}r_1^2-d_{55}r_3^2}\frac{\partial^2\chi}{\partial z^2}, \end{aligned}\right\} \tag{6·18}$$

where
$$r_1\nabla_1^2\chi + r_3\frac{\partial^2\chi}{\partial z^2} = 0, \quad \frac{\partial^2\chi}{\partial z^2} = T', \tag{6·19}$$

and
$$m = \frac{r_3}{r_1}l = \frac{(\omega_3-\omega_1)\,r_3r_1}{2(d_{44}r_1^2-d_{55}r_3^2)}. \tag{6·20}$$

When, in addition, k_1 and k_2 are equal so that

$$\frac{r_3}{r_1} = k_1 = k_2, \quad k_1^2 = \frac{d_{44}}{d_{55}}, \tag{6·21}$$

the particular solution (6·18) breaks down and a particular solution takes the form

$$\left.\begin{aligned} u &= lz^2\frac{\partial^3\chi}{\partial x\,\partial z^2}, \\ v &= lz^2\frac{\partial^3\chi}{\partial y\,\partial z^2}, \\ w &= mz^2\frac{\partial^3\chi}{\partial z^3} - 2mz\frac{\partial^2\chi}{\partial z^2} + 2m\frac{\partial\chi}{\partial z}, \\ p' &= \frac{(\omega_1-\omega_3)}{8r_1d_{44}}\{d_{44}r_1+(d_{55}-a)\,r_3\}\,z^2\frac{\partial^4\chi}{\partial z^4} + \tfrac{1}{2}(\omega_1-\omega_3)\,z\frac{\partial^3\chi}{\partial z^3} - \tfrac{1}{4}(3\omega_1+\omega_3)\frac{\partial^2\chi}{\partial z^2}, \end{aligned}\right\} \tag{6·22}$$

where
$$m = \frac{r_3}{r_1}l = \frac{(\omega_3-\omega_1)\,r_3}{8d_{44}r_1}. \tag{6·23}$$

Similarly, in the second case, if the roots k_1, k_2 of the quadratic (6·10) are equal, the general solution to the homogeneous problem becomes

$$\left.\begin{aligned} u &= \frac{\partial\chi_1}{\partial x} + z\frac{\partial^2\chi_2}{\partial x\,\partial z} + \frac{\partial\chi_3}{\partial y}, \\ v &= \frac{\partial\chi_1}{\partial y} + z\frac{\partial^2\chi_2}{\partial y\,\partial z} - \frac{\partial\chi_3}{\partial x}, \\ w &= k_1\frac{\partial\chi_1}{\partial z} + k_1 z\frac{\partial^2\chi_2}{\partial z^2} - k_1\frac{\partial\chi_2}{\partial z}, \\ p' &= -\{d_{44}+(d_{55}-a)\,k_1\}\left(\frac{\partial^2\chi_1}{\partial z^2} + z\frac{\partial^3\chi_2}{\partial z^3}\right) - 2d_{44}\frac{\partial^2\chi_2}{\partial z^2}, \end{aligned}\right\} \tag{6·24}$$

where the potential functions χ_i satisfy (6·14).

† Once again we remark that these are not the only cases of degeneracy but that, in most cases, the solution can be found by a limiting process from the solution in the general case.

Thus the general solution to the incompressibility condition (6·5) and the equations of equilibrium (6·6) is given by adding the particular solution (6·7), or (6·18), or (6·22) to the general solution (6·17), or (6·24).

The general solution in the non-degenerate case is given by displacements of the form (5·59) where χ_i, χ, satisfy (6·14) and (6·8) respectively, k_1, k_2, l, m follow from (6·10) and (6·9), and

$$p' = (d_{44}-c+d_{55}k_1)\,k_1\frac{\partial^2\chi_1}{\partial z^2}+(d_{44}-c+d_{55}k_2)\,k_2\frac{\partial^2\chi_2}{\partial z^2} - \frac{r_3\omega_1[r_1(d_{44}-c)+r_3d_{55}]+r_1\omega_3[r_1d_{44}+r_3(d_{55}-a)]}{d_{55}(r_1k_1-r_3)\,(r_1k_2-r_3)}\frac{\partial^2\chi}{\partial z^2}. \tag{6·25}$$

7. Crack problems: compressible medium

The theory developed in §§ 5, 6 is now used to solve two problems. The first is concerned with a penny-shaped crack in an infinite medium and the second in § 9 with boundary-value problems in the half space. Throughout this and the next sections we use the method given by Green & Zerna (1954, p. 175; see also Collins 1959) of representing potential functions as particular integrals but the results could also be obtained with the aid of Hankel transforms. Hankel transforms have been used to solve analogous problems in classical elasticity by Olesiak & Sneddon (1960).

Suppose that an infinite isotropic medium is deformed as in §§ 5, 6 but that, in addition, the stress component τ^{33} is zero so that

$$\Phi\lambda_3^2+2\Psi\lambda_3^2\lambda_1^2+p = 0. \tag{7·1}$$

Further, there is a penny-shaped crack of radius a^* in the plane $z = 0$ and before the initial finite deformation this was a crack of radius a_0 in the same plane, where

$$a^* = \lambda_1 a_0. \tag{7·2}$$

The presence of the crack does not affect the stress distribution since τ^{33} vanishes. We use cylindrical polar† co-ordinates (r, θ, z) in the deformed medium so that the crack is defined by

$$z = 0, \quad 0 \leqslant r \leqslant a^*. \tag{7·3}$$

We suppose that equal (small) distributions of temperature $\epsilon f(r)$ and pressure $\epsilon g(r)$ are applied to both the upper and lower surfaces of the crack. This problem is equivalent to the axially symmetric problem for a semi-infinite medium $z \geqslant 0$, previously deformed as in §§ 5, 6 with the surface $z = 0$ free from applied stress, and then subjected to the following boundary conditions:

$$\left.\begin{aligned} T'(r,z) &= f(r) \quad && (z=0,\ 0\leqslant r\leqslant a^*),\\ \frac{\partial T'(r,z)}{\partial z} &= 0 && (z=0,\ a^*<r),\end{aligned}\right\} \tag{7·4}$$

$$\left.\begin{aligned} \tau'^{33} &= g(r) \quad && (z=0,\ 0\leqslant r\leqslant a^*),\\ w &= 0 && (z=0,\ a^*<r),\\ \tau'^{13} = \tau'^{23} &= 0 && (z=0,\ 0\leqslant r<\infty).\end{aligned}\right\} \tag{7·5}$$

† The constant r defined in (5·18) is not used here so there should be no risk of confusion.

We assume that $f(r)$, $g(r)$ are sectionally continuous in $0 \leqslant r \leqslant a^*$ with at most a finite number of finite discontinuities.

We consider first the compressible medium of § 5, and deal only with the non-degenerate case, since the solutions in the other cases may be found from this by a limiting process. From (5·59), (4·5) and (4·6), and remembering (5·11), (5·12), (5·40) and (5·46), we have

$$\tau'^{13} = c_{44}\frac{\partial}{\partial x}\left[(1+k_1)\frac{\partial \chi_1}{\partial z}+(1+k_2)\frac{\partial \chi_2}{\partial z}+(l+m)\frac{\partial \chi}{\partial z}\right]+c_{44}\frac{\partial^2 \chi_3}{\partial y\,\partial z}, \tag{7·6}$$

$$\tau'^{23} = c_{44}\frac{\partial}{\partial y}\left[(1+k_1)\frac{\partial \chi_1}{\partial z}+(1+k_2)\frac{\partial \chi_2}{\partial z}+(l+m)\frac{\partial \chi}{\partial z}\right]-c_{44}\frac{\partial^2 \chi_3}{\partial x\,\partial z}, \tag{7·7}$$

$$\tau'^{33} = (c_{33}k_1-c_{31}\nu_1)\frac{\partial^2 \chi_1}{\partial z^2}+(c_{33}k_2-c_{31}\nu_2)\frac{\partial^2 \chi_2}{\partial z^2}+(c_{33}m-c_{31}lr_3/r_1)\frac{\partial^2 \chi}{\partial z^2}+\omega_3 T'. \tag{7·8}$$

The boundary conditions (7·5) can be satisfied if

$$\chi_3 \equiv 0,$$

and
$$(1+k_1)\frac{\partial \chi_1}{\partial z}+(1+k_2)\frac{\partial \chi_2}{\partial z}+(l+m)\frac{\partial \chi}{\partial z} = 0 \quad (z=0,\ 0\leqslant r<\infty), \tag{7·9}$$

$$k_1\frac{\partial \chi_1}{\partial z}+k_2\frac{\partial \chi_2}{\partial z}+m\frac{\partial \chi}{\partial z} = 0 \quad (z=0,\ a^*<r), \tag{7·10}$$

$$(c_{33}k_1-c_{31}\nu_1)\frac{\partial^2 \chi_1}{\partial z^2}+(c_{33}k_2-c_{31}\nu_2)\frac{\partial^2 \chi_2}{\partial z^2}+(c_{33}m-c_{31}lr_3/r_1)\frac{\partial^2 \chi}{\partial z^2}+\omega_3 T' = g(r) \quad (z=0,\ 0\leqslant r\leqslant a^*). \tag{7·11}$$

We recall that χ is a particular integral of the equations

$$r_1\nabla_1^2\chi+r_3\frac{\partial^2 \chi}{\partial z^2} = 0, \quad \frac{\partial^2 \chi}{\partial z^2} = T', \tag{7·12}$$

and, in view of equations (5·44) satisfied by χ_1 and χ_2, we now put

$$\left.\begin{aligned} \chi_1 &= \alpha\chi\left\{r,\ z\left(\frac{r_3}{r_1\nu_1}\right)^{\frac{1}{2}}\right\}+\frac{\surd\nu_1}{1+k_1}\phi\left(r,\ \frac{z}{\surd\nu_1}\right),\\ \chi_2 &= \beta\chi\left\{r,\ z\left(\frac{r_3}{r_1\nu_2}\right)^{\frac{1}{2}}\right\}-\frac{\surd\nu_2}{1+k_2}\phi\left(r,\ \frac{z}{\surd\nu_2}\right),\end{aligned}\right\} \tag{7·13}$$

and choose the constants α, β so that the boundary conditions (7·9) and (7·10) are satisfied identically as far as the terms in $\partial\chi/\partial z$ are concerned. Thus

$$\alpha\left(\frac{r_3}{r_1\nu_1}\right)^{\frac{1}{2}} = \frac{lk_2-m}{k_1-k_2}, \quad \beta\left(\frac{r_3}{r_1\nu_2}\right)^{\frac{1}{2}} = \frac{m-lk_1}{k_1-k_2}. \tag{7·14}$$

Also, from (5·44), the function $\phi(r,z)$ satisfies the equation

$$\nabla^2\phi = 0 \tag{7·15}$$

and we observe that the boundary condition (7·9) is completely satisfied by the values (7·13) for χ_1 and χ_2. Using (7·13) to (7·15) we see that the remaining boundary conditions (7·10) and (7·11) reduce to

$$\left.\begin{aligned} \frac{\partial^2\phi(r,z)}{\partial z^2} &= Xg(r)-Yf(r) \quad &(z=0,\ 0\leqslant r\leqslant a^*),\\ \frac{\partial \phi}{\partial z} &= 0 &(z=0,\ a^*<r),\end{aligned}\right\} \tag{7·16}$$

where
$$\left.\begin{aligned}\frac{1}{X}&=\frac{c_{33}k_1-c_{31}\nu_1}{(1+k_1)\sqrt{\nu_1}}-\frac{c_{33}k_2-c_{31}\nu_2}{(1+k_2)\sqrt{\nu_2}},\\ \frac{Y}{X}&=c_{33}m-c_{31}\frac{lr_3}{r_1}+\omega_3+\frac{(c_{33}k_1-c_{31}\nu_1)\,\alpha r_3}{r_1\nu_1}+\frac{(c_{33}k_2-c_{31}\nu_2)\,\beta r_3}{r_1\nu_2}.\end{aligned}\right\} \tag{7·17}$$

We consider first equations (7·12) subject to the conditions (7·4). Since T' satisfies the same differential equation as χ it can be shown (see, for example, Green & Zerna 1954, p. 175) that a suitable integral for T' which satisfies the second condition in (7·4), and vanishes at infinity, is

$$T'(r,z)=\frac{1}{2}\int_{-a^*}^{a^*}\frac{h(t)\,\mathrm{d}t}{\{r^2+(\zeta+\mathrm{i}t)^2\}^{\frac{1}{2}}},\quad \zeta=z\left(\frac{r_1}{r_3}\right)^{\frac{1}{2}}, \tag{7·18}$$

where $h(t)$ is a real sectionally continuous even function of t and where, for $0\leqslant t\leqslant a^*$,

$$\left.\begin{aligned}&\{r^2+(\zeta+\mathrm{i}t)^2\}^{\frac{1}{2}}=\xi\,\mathrm{e}^{\frac{1}{2}\mathrm{i}\eta},\quad \{r^2+(\zeta-\mathrm{i}t)^2\}^{\frac{1}{2}}=\xi\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}\eta}\quad(\xi\geqslant 0),\\ &\xi^2\cos\eta=r^2+\zeta^2-t^2,\quad \xi^2\sin\eta=2\zeta t\quad(0\leqslant\eta\leqslant\pi).\end{aligned}\right\} \tag{7·19}$$

If we observe that when $\zeta=0$

$$\left.\begin{aligned}&\{r^2+(\zeta+\mathrm{i}t)^2\}^{\frac{1}{2}}=\{r^2+(\zeta-\mathrm{i}t)^2\}^{\frac{1}{2}}=(r^2-t^2)^{\frac{1}{2}}\quad(r\geqslant t),\\ &\{r^2+(\zeta+\mathrm{i}t)^2\}^{\frac{1}{2}}=-\{r^2+(\zeta-\mathrm{i}t)^2\}^{\frac{1}{2}}=\mathrm{i}(t^2-r^2)^{\frac{1}{2}}\quad(r\leqslant t),\end{aligned}\right\} \tag{7·20}$$

the first condition in (7·4) gives

$$\int_0^r\frac{h(t)\,\mathrm{d}t}{(r^2-t^2)^{\frac{1}{2}}}=f(r)\quad(0\leqslant r\leqslant a^*). \tag{7·21}$$

This integral equation can be solved by elementary analysis to yield

$$h(t)=\frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\frac{rf(r)\,\mathrm{d}r}{(t^2-r^2)^{\frac{1}{2}}}\quad(t\geqslant 0). \tag{7·22}$$

We can now verify that $h(t)$ is sectionally continuous.

When T' is given by (7·18) equations (7·12) must be solved for χ. The solution is undetermined to the extent of an additive function of the form $\zeta\omega(x,y)+\Omega(x,y)$, where ω, Ω are plane harmonic functions. The indeterminacy can be removed by imposing the condition that all stress components must vanish at infinity so that we require that all second-order derivatives of χ vanish at infinity. The function χ and its first derivatives will not, in general, vanish at infinity, which implies that the components of displacement may not vanish at infinity.

From (7·12) and (7·18), remembering the conditions imposed at infinity, we get

$$\left(\frac{r_1}{r_3}\right)^{\frac{1}{2}}\frac{\partial\chi(r,z)}{\partial z}=\frac{1}{2}\int_{-a^*}^{a^*}h(t)\ln\left[\zeta+\mathrm{i}t+\{r^2+(\zeta+\mathrm{i}t)^2\}^{\frac{1}{2}}\right]\mathrm{d}t, \tag{7·23}$$

where here and subsequently, for definiteness, the principal value of the logarithm is taken. Similarly

$$\frac{r_1}{r_3}\chi(r,z)=\frac{1}{2}\int_{-a^*}^{a^*}h(t)\Big[(\zeta+\mathrm{i}t)\ln\left[\zeta+\mathrm{i}t+\{r^2+(\zeta+\mathrm{i}t)^2\}^{\frac{1}{2}}\right]-\{r^2+(\zeta+\mathrm{i}t)^2\}^{\frac{1}{2}}\Big]\,\mathrm{d}t. \tag{7·24}$$

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We now consider equations (7·15) and (7·16). If $\partial\phi/\partial z$ vanishes at infinity then a suitable value for $\partial\phi/\partial z$ which satisfies (7·15) and the second condition in (7·16) is

$$\frac{\partial\phi(r,z)}{\partial z}=\frac{1}{2\mathrm{i}}\int_{-a^*}^{a^*}\frac{q(t)\,\mathrm{d}t}{\{r^2+(z+\mathrm{i}t)^2\}^{\frac{1}{2}}},\tag{7·25}$$

where $q(t)$ is a real sectionally continuous odd function of t. The verification of the second condition in (7·16) is immediate if we remember (7·20). From (7·25)

$$\frac{\partial^2\phi}{\partial z^2}=\frac{1}{2\mathrm{i}r}\frac{\partial}{\partial r}\int_{-a^*}^{a^*}\frac{(z+\mathrm{i}t)\,q(t)\,\mathrm{d}t}{\{r^2+(z+\mathrm{i}t)^2\}^{\frac{1}{2}}}.\tag{7·26}$$

If we again use (7·20) we see that the first condition in (7·16) is satisfied if

$$\frac{\partial}{\partial r}\int_0^r\frac{tq(t)\,\mathrm{d}t}{(r^2-t^2)^{\frac{1}{2}}}=r[Xg(r)-Yf(r)]\quad(0\leqslant r\leqslant a^*).\tag{7·27}$$

Alternatively an integration with respect to r yields

$$\int_0^r\frac{tq(t)\,\mathrm{d}t}{(r^2-t^2)^{\frac{1}{2}}}=\int_0^r u\,[Xg(u)-Yf(u)]\,\mathrm{d}u\quad(0\leqslant r\leqslant a^*),\tag{7·28}$$

the constant of integration vanishing in view of the conditions imposed on $f(u)$, $g(u)$, $q(t)$. Hence

$$q(t)=\frac{2}{\pi}\int_0^t\frac{r[Xg(r)-Yf(r)]\,\mathrm{d}r}{(t^2-r^2)^{\frac{1}{2}}}\quad(t\geqslant 0),\tag{7·29}$$

which satisfies the conditions imposed on $q(t)$. Also, if all derivatives of ϕ with respect to r or z are zero at infinity, and we recall (7·15), equation (7·25) can be integrated to give

$$\phi(r,z)=\frac{1}{2\mathrm{i}}\int_{-a^*}^{a^*}q(t)\ln[z+\mathrm{i}t+\{r^2+(z+\mathrm{i}t)^2\}^{\frac{1}{2}}]\,\mathrm{d}t.\tag{7·30}$$

For some purposes it is convenient to write the solution in an alternative form. From (7·29) we see that $q(t)$ is a linear combination of a term depending only on the specified temperature distribution and a term depending only on the given pressure distribution. From (7·22) and (7·29) we have

$$q(t)=Xs(t)-YH(t),\tag{7·31}$$

where

$$H(t)=\frac{2}{\pi}\int_0^t\frac{rf(r)\,\mathrm{d}r}{(t^2-r^2)^{\frac{1}{2}}},\quad h(t)=H'(t),\tag{7·32}$$

and

$$s(t)=\frac{2}{\pi}\int_0^t\frac{rg(r)\,\mathrm{d}r}{(t^2-r^2)^{\frac{1}{2}}}.\tag{7·33}$$

With the help of (7·24) and (7·31) and an integration by parts, equation (7·30) becomes

$$\begin{aligned}\phi(r,z)={}&X\psi(r,z)-Y\frac{r_1}{r_3}\chi\left\{r,z\left(\frac{r_3}{r_1}\right)^{\frac{1}{2}}\right\}\\&+\frac{YH(a^*)}{2}(z+\mathrm{i}a^*)\ln[z+\mathrm{i}a^*+\{r^2+(z+\mathrm{i}a^*)^2\}^{\frac{1}{2}}]\\&+\frac{YH(a^*)}{2}(z-\mathrm{i}a^*)\ln[z-\mathrm{i}a^*+\{r^2+(z-\mathrm{i}a^*)^2\}^{\frac{1}{2}}]\\&-\frac{YH(a^*)}{2}\{r^2+(z+\mathrm{i}a^*)^2\}^{\frac{1}{2}}-\frac{YH(a^*)}{2}\{r^2+(z-\mathrm{i}a^*)^2\}^{\frac{1}{2}},\end{aligned}\tag{7·34}$$

where
$$\psi(r,z) = \frac{1}{2\mathrm{i}}\int_{-a^*}^{a^*} s(t)\ln[z+\mathrm{i}t+\{r^2+(z+\mathrm{i}t)^2\}^{\frac{1}{2}}]\,\mathrm{d}t. \tag{7·35}$$

The solution (7·13) is now

$$\chi_1 = \left[\alpha - \frac{Yr_1\nu_1^{\frac{1}{2}}}{r_3(1+k_1)}\right]\chi\left\{r, z\left(\frac{r_3}{r_1\nu_1}\right)^{\frac{1}{2}}\right\} + \frac{X\nu_1^{\frac{1}{2}}}{1+k_1}\psi\left(r,\frac{z}{\nu_1^{\frac{1}{2}}}\right)$$
$$+\frac{Y\nu_1^{\frac{1}{2}}H(a^*)}{2(1+k_1)}\Big[(z/\nu_1^{\frac{1}{2}}+\mathrm{i}a^*)\ln[z/\nu_1^{\frac{1}{2}}+\mathrm{i}a^*+\{r^2+(z/\nu_1^{\frac{1}{2}}+\mathrm{i}a^*)^2\}^{\frac{1}{2}}]$$
$$+(z/\nu_1^{\frac{1}{2}}-\mathrm{i}a^*)\ln[z/\nu_1^{\frac{1}{2}}-\mathrm{i}a^*+\{r^2+(z/\nu_1^{\frac{1}{2}}-\mathrm{i}a^*)^2\}^{\frac{1}{2}}]$$
$$-\{r^2+(z/\nu_1^{\frac{1}{2}}+\mathrm{i}a^*)^2\}^{\frac{1}{2}}-\{r^2+(z/\nu_1^{\frac{1}{2}}-\mathrm{i}a^*)^2\}^{\frac{1}{2}}\Big], \tag{7·36}$$

$$\chi_2 = \left[\beta + \frac{Yr_1\nu_2^{\frac{1}{2}}}{r_3(1+k_2)}\right]\chi\left\{r, z\left(\frac{r_3}{r_1\nu_2}\right)^{\frac{1}{2}}\right\} - \frac{X\nu_2^{\frac{1}{2}}}{1+k_2}\psi(r, z/\nu_2^{\frac{1}{2}})$$
$$-\frac{Y\nu_2^{\frac{1}{2}}H(a^*)}{2(1+k_2)}\Big[(z/\nu_2^{\frac{1}{2}}+\mathrm{i}a^*)\ln[z/\nu_2^{\frac{1}{2}}+\mathrm{i}a^*+\{r^2+(z/\nu_2^{\frac{1}{2}}+\mathrm{i}a^*)^2\}^{\frac{1}{2}}]$$
$$+(z/\nu_2^{\frac{1}{2}}-\mathrm{i}a^*)\ln[z/\nu_2^{\frac{1}{2}}-\mathrm{i}a^*+\{r^2+(z/\nu_2^{\frac{1}{2}}-\mathrm{i}a^*)^2\}^{\frac{1}{2}}]$$
$$-\{r^2+(z/\nu_2^{\frac{1}{2}}+\mathrm{i}a^*)^2\}^{\frac{1}{2}}-\{r^2+(z/\nu_2^{\frac{1}{2}}-\mathrm{i}a^*)^2\}^{\frac{1}{2}}\Big], \tag{7·37}$$

where χ and ψ are given by (7·24) and (7·35) and depend respectively only on the applied temperature and pressure on the crack.

The problem of the crack with boundary conditions of the form (7·4) and (7·5) is now solved, and corresponding stresses and displacements may be found by using the formulae of § 5. It is, however, possible to obtain some useful information without going through the complete analysis in a particular problem. For example, the normal displacement at the surface of the crack is, from (5·59), given by

$$w = k_1\frac{\partial\chi_1}{\partial z}+k_2\frac{\partial\chi_2}{\partial z}+m\frac{\partial\chi}{\partial z} \quad (z=0,\ 0\leqslant r\leqslant a^*\ , \tag{7·38}$$

where χ_1, χ_2 are obtained from (7·13) and χ from (7·24). Hence, remembering (7·14), we have

$$w = \left(\frac{k_1}{1+k_1}-\frac{k_2}{1+k_2}\right)\frac{\partial\phi(r,z)}{\partial z} \quad (z=0,\ 0\leqslant r\leqslant a^*), \tag{7·39}$$

where, from (7·25) and (7·20),

$$\frac{\partial\phi(r,z)}{\partial z} = -\int_r^{a^*}\frac{q(t)\,\mathrm{d}t}{(t^2-r^2)^{\frac{1}{2}}} \quad (z=0,\ 0\leqslant r\leqslant a^*). \tag{7·40}$$

Also $q(t)$ is given by (7·29).

In the simple case when a constant normal pressure and a constant temperature are applied to the surfaces of the crack, so that

$$f(r) = f_0, \quad g(r) = -g_0, \tag{7·41}$$

where f_0, g_0 are constants, equation (7·29) at once yields

$$q(t) = -\frac{2}{\pi}(Xg_0+Yf_0)\,t. \tag{7·42}$$

The normal displacement at the crack then becomes

$$w = \frac{2}{\pi}(Xg_0+Yf_0)\left(\frac{k_1}{1+k_1}-\frac{k_2}{1+k_2}\right)(a^{*2}-r^2)^{\frac{1}{2}} \quad (0\leqslant r\leqslant a^*). \tag{7·43}$$

8. Crack problems: incompressible medium

The problem specified at the beginning of § 7 is now solved for an incompressible medium using the theory of § 6. The general non-degenerate solution is given by the sum of the two solutions (6·7) and (6·17) so that

$$\left.\begin{aligned} u &= \frac{\partial \chi_1}{\partial x}+\frac{\partial \chi_2}{\partial x}+\frac{\partial \chi_3}{\partial y}+l\frac{\partial \chi}{\partial x}, \\ v &= \frac{\partial \chi_1}{\partial y}+\frac{\partial \chi_2}{\partial y}-\frac{\partial \chi_3}{\partial x}+l\frac{\partial \chi}{\partial y}, \\ w &= k_1\frac{\partial \chi_1}{\partial z}+k_2\frac{\partial \chi_2}{\partial z}+m\frac{\partial \chi}{\partial z}, \end{aligned}\right\} \tag{8·1}$$

$$\begin{aligned} p' = {} & (d_{44}-c+d_{55}k_1)\,k_1\frac{\partial^2 \chi_1}{\partial z^2}+(d_{44}-c+d_{55}k_2)\,k_2\frac{\partial^2 \chi_2}{\partial z^2} \\ & -\frac{r_3\omega_1[r_1(d_{44}-c)+r_3 d_{55}]+r_1\omega_3[r_1 d_{44}+r_3(d_{55}-a)]}{d_{55}(r_1k_1-r_3)\,(r_1k_2-r_3)}\frac{\partial^2 \chi}{\partial z^2}. \end{aligned} \tag{8·2}$$

From (4·6) and (8·1), since $c_{44} = c_{55}$ in our problem,

$$\left.\begin{aligned} \tau'^{13} &= c_{44}\frac{\partial}{\partial x}\left[(1+k_1)\frac{\partial \chi_1}{\partial z}+(1+k_2)\frac{\partial \chi_2}{\partial z}+(l+m)\frac{\partial \chi}{\partial z}\right]+c_{44}\frac{\partial^2 \chi_3}{\partial y\,\partial z}, \\ \tau'^{23} &= c_{44}\frac{\partial}{\partial y}\left[(1+k_1)\frac{\partial \chi_1}{\partial z}+(1+k_2)\frac{\partial \chi_2}{\partial z}+(l+m)\frac{\partial \chi}{\partial z}\right]-c_{44}\frac{\partial^2 \chi_3}{\partial x\,\partial z}. \end{aligned}\right\} \tag{8·3}$$

Also, since $\tau^{33} = 0$ here, it follows that $\tau'^{33} = t'^{33}$ so that, from (6·2) and (8·1),

$$\tau'^{33} = (d_{44}+d_{55}k_1)\,k_1\frac{\partial^2 \chi_1}{\partial z^2}+(d_{44}+d_{55}k_2)\,k_2\frac{\partial^2 \chi_2}{\partial z^2}+(d_{44}+d_{55}r_3/r_1)\,m\frac{\partial^2 \chi}{\partial z^2}, \tag{8·4}$$

if we make use of (6·9) and (6·10).

The boundary conditions (7·4) and (7·5) can be satisfied if

$$\chi_3 \equiv 0, \tag{8·5}$$

$$(1+k_1)\frac{\partial \chi_1}{\partial z}+(1+k_2)\frac{\partial \chi_2}{\partial z}+(l+m)\frac{\partial \chi}{\partial z} = 0 \quad (z = 0,\ 0 \leqslant r \leqslant \infty), \tag{8·6}$$

$$k_1\frac{\partial \chi_1}{\partial z}+k_2\frac{\partial \chi_2}{\partial z}+m\frac{\partial \chi}{\partial z} = 0 \quad (z = 0,\ a^* < r), \tag{8·7}$$

$$\begin{aligned} & (d_{44}+d_{55}k_1)\,k_1\frac{\partial^2 \chi_1}{\partial z^2}+(d_{44}+d_{55}k_2)\,k_2\frac{\partial^2 \chi_2}{\partial z^2} \\ & \qquad +(d_{44}+d_{55}r_3/r_1)\,m\frac{\partial^2 \chi}{\partial z^2} = g(r) \quad (z = 0,\ 0 \leqslant r \leqslant a^*). \end{aligned} \tag{8·8}$$

Following the same development as in § 7, these conditions can be satisfied by

$$\left.\begin{aligned} \chi_1 &= \alpha\chi\left\{r, z\left(\frac{r_3}{r_1k_1}\right)^{\frac{1}{2}}\right\}+\frac{\sqrt{k_1}}{1+k_1}\phi\left(r, \frac{z}{\sqrt{k_1}}\right), \\ \chi_2 &= \beta\chi\left\{r, z\left(\frac{r_3}{r_1k_2}\right)^{\frac{1}{2}}\right\}-\frac{\sqrt{k_2}}{1+k_2}\phi\left(r, \frac{z}{\sqrt{k_2}}\right), \end{aligned}\right\} \tag{8·9}$$

where α, β are not the same as in § 7 but are given by

$$\alpha\left(\frac{r_3}{r_1 k_1}\right)^{\frac{1}{2}} = \frac{lk_2 - m}{k_1 - k_2}, \quad \beta\left(\frac{r_3}{r_1 k_2}\right)^{\frac{1}{2}} = \frac{m - lk_1}{k_1 - k_2}. \tag{8·10}$$

Also

$$\nabla^2\phi(r,z) = 0, \tag{8·11}$$

$$\left.\begin{aligned} \frac{\partial^2\phi(r,z)}{\partial z^2} &= Xg(r) - Yf(r) \quad (z=0,\ 0 \leqslant r \leqslant a^*), \\ \frac{\partial\phi(r,z)}{\partial z} &= 0 \qquad\qquad\qquad\quad (z=0,\ a^* < r), \end{aligned}\right\} \tag{8·12}$$

the constants X, Y being different from those in § 7 and now defined by

$$\left.\begin{aligned} \frac{1}{X} &= \frac{(d_{44}+d_{55}k_1)\,k_1^{\frac{1}{2}}}{1+k_1} - \frac{(d_{44}+d_{55}k_2)\,k_2^{\frac{1}{2}}}{1+k_2}, \\ \frac{Y}{X} &= (d_{44}+d_{55}k_1)\,\alpha r_3/r_1 + (d_{44}+d_{55}k_2)\,\beta r_3/r_1 + (d_{44}+d_{55}r_3/r_1)\,m. \end{aligned}\right\} \tag{8·13}$$

The function χ still satisfies equations (7·12) and T' has the value (7·18).

The problem has been reduced to a boundary-value problem of the same form as that solved in § 7 and results can at once be written down from those in that section. We therefore omit further details but we observe that the normal displacement at the crack is still given by (7·39), (7·40) and (7·29) with appropriate values (8·13) for X and Y and where k_1, k_2 are now the roots of equation (6·10).

A particular case of some interest arises when the incompressible material has negligible internal energy so that the Helmholtz function is proportional to the temperature T. A further special case of this occurs for a Mooney type material in which the energy W is

$$W = A_1 T(I_1 - 3) + A_2 T(I_2 - 3), \tag{8·14}$$

A_1, A_2 being constants. When $A_2 = 0$ the form (8·14) has been found to be of value for vulcanized rubber (see Treloar (1958) for further details). Most of the constants used in § 6 have been evaluated in Green & Zerna (1954, p. 135) when T is constant but are repeated here with the slight change of notation. The further constants arising from the presence of T in (8·14) are also included. Thus

$$\left.\begin{aligned} \Phi = 2A_1 T, \quad \Psi &= 2A_2 T, \\ A = B &= F = 0, \\ L = 2A_1, \quad M &= 2A_2, \end{aligned}\right\} \tag{8·15}$$

$$a = 2d_{55} = 4\lambda_1^2(A_1 + A_2\lambda_1^2)\,T, \quad c = 2d_{44} = 4\lambda_3^2(A_1 + A_2\lambda_1^2)\,T, \tag{8·16}$$

the temperature T in (8·15) being constant. Also

$$\left.\begin{aligned} k_1 = 1, \quad k_2 &= \lambda_3^3 = 1/\lambda_1^6, \\ \frac{k_1}{1+k_1} - \frac{k_2}{1+k_2} &= \frac{\lambda_1^6 - 1}{2(\lambda_1^6+1)}, \end{aligned}\right\} \tag{8·17}$$

and

$$\left.\begin{aligned} \omega_1 &= 2A_1\lambda_1^2 + 2A_2\lambda_1^2(\lambda_1^2 + \lambda_3^2), \\ \omega_3 &= 2A_1\lambda_3^2 + 4A_2\lambda_3, \\ \frac{\omega_3 - \omega_1}{d_{55}} &= \left(\frac{1}{\lambda_1^6} - 1\right)\frac{1}{T}. \end{aligned}\right\} \tag{8·18}$$

It follows from (6·9), (8·17) and (8·18) that

$$m = \frac{r_3 l}{r_1} = \frac{(1-\lambda_1^6)\, r_1 r_3}{(r_1-r_3)\,(r_1-r_3\lambda_1^6)\, T}, \tag{8·19}$$

and then, from (8·10),

$$\alpha\left(\frac{r_3}{r_1}\right)^{\frac{1}{2}} = \frac{r_1}{(r_3-r_1)\, T}, \quad \beta\left(\frac{r_3}{r_1}\right)^{\frac{1}{2}} = \frac{\lambda_1^3 r_1}{(r_1-r_3\lambda_1^6)\, T}. \tag{8·20}$$

Further, from (8·13) we have,

$$X = \frac{\lambda_1^4(\lambda_1^6+1)}{T(A_1+A_2\lambda_1^2)\,(\lambda_1^3-1)\,(\lambda_1^9+\lambda_1^6+3\lambda_1^3-1)}, \tag{8·21}$$

$$Y = \frac{2(1+\lambda_1^6)\,[(1-\lambda_1^3)\, r_1^{\frac{1}{2}} - \lambda_1^3(1+\lambda_1^3)\, r_3^{\frac{1}{2}}]\, r_3^{\frac{1}{2}}}{T(r_1^{\frac{1}{2}}+r_3^{\frac{1}{2}})\,(r_1^{\frac{1}{2}}+\lambda_1^3 r_3^{\frac{1}{2}})\,(\lambda_1^9+\lambda_1^6+3\lambda_1^3-1)}, \tag{8·22}$$

and, finally, we recall from (4·19) that

$$\left.\begin{aligned} r_1 &= \mathscr{C}_1 + \tfrac{1}{2}\mathscr{C}_2(\lambda_1^2-1) + \tfrac{1}{4}\mathscr{C}_3(\lambda_1^2-1)^2 \geqslant 0, \\ r_3 &= \mathscr{C}_1 + \tfrac{1}{2}\mathscr{C}_2(\lambda_3^2-1) + \tfrac{1}{4}\mathscr{C}_3(\lambda_3^2-1)^2 \geqslant 0, \end{aligned}\right\} \tag{8·23}$$

where $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$ are polynomials in the invariants I_1-3, I_2-3, i.e. in the two quantities

$$\frac{(\lambda_1^2-1)^2\,(1+2\lambda_1^2)}{\lambda_1^4}, \quad \frac{(\lambda_1^2-1)^2\,(2+\lambda_1^2)}{\lambda_1^2}. \tag{8·24}$$

From (8·17), (8·21) and (8·22) we see that the value of the normal displacement becomes infinite when λ_1 is a root of the equation

$$\lambda_1^9+\lambda_1^6+3\lambda_1^3-1 = 0,$$

which has a root near $\lambda_1 = \frac{2}{3}$ suggesting that for this value of λ_1 the original finite deformation is unstable, as indicated in Green & Zerna (1954, p. 138).

9. Half-space problems

Some mixed boundary-value problems associated with the half space can be discussed using a technique similar to that for the crack problem of §§ 7, 8. Axially symmetric problems in which the surface values of the normal pressure and the temperature are prescribed everywhere along the boundary, together with zero surface shears, can be solved with the help of Hankel transforms. Alternatively, results can be deduced from the analysis of the previous §§ 7, 8 by a limiting process. To illustrate this we consider a special problem which is a generalization of that discussed at the end of § 7. We assume that constant normal pressure and temperature are applied over a circular area of both surfaces of the crack of radius $a_1 \leqslant a^*$. Over the remainder of the crack the pressure and temperature is zero. Thus

$$\left.\begin{aligned} f(r) &= f_0, \quad g(r) = -g_0 \quad (0 \leqslant r \leqslant a_1), \\ f(r) &= 0, \quad g(r) = 0 \quad (a_1 < r \leqslant a^*). \end{aligned}\right\} \tag{9·1}$$

From (7·29) we then have

$$\left.\begin{aligned} q(t) &= -\frac{2}{\pi}(Xg_0+Yf_0) && (0 \leqslant t \leqslant a_1), \\ q(t) &= -\frac{2}{\pi}(Xg_0+Yf_0)\,[t-(t^2-a_1^2)^{\frac{1}{2}}] && (a_1 \leqslant t \leqslant a^*). \end{aligned}\right\} \tag{9·2}$$

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The normal displacement at the surface of the crack is then obtained from (7·39) and (7·40) in the form

$$w = \frac{2}{\pi}(Xg_0 + Yf_0)\left(\frac{k_1}{1+k_1} - \frac{k_2}{1+k_2}\right) w_1, \tag{9·3}$$

where

$$\left.\begin{aligned} w_1 &= (a_1^2 - r^2)^{\frac{1}{2}} + \int_{a_1}^{a^*} \frac{t-(t^2-a_1^2)^{\frac{1}{2}}}{(t^2-r^2)^{\frac{1}{2}}}\,\mathrm{d}t \quad (0 \leqslant r \leqslant a_1),\\ w_1 &= \int_r^{a^*} \frac{t-(t^2-a_1^2)^{\frac{1}{2}}}{(t^2-r^2)^{\frac{1}{2}}}\,\mathrm{d}t \quad (a_1 \leqslant r \leqslant a^*).\end{aligned}\right\} \tag{9·4}$$

If $a_1 = a^*$ we recover the formula (7·43). Otherwise part of the integrals in (9·4) can be expressed in terms of elliptic functions.

We now let the radius a^* of the crack tend to infinity. We then have a boundary-value problem for the half space in which the normal pressure and temperature are prescribed on $z = 0$, being zero outside a circle of radius a_1, the shear stresses on $z = 0$ being zero. The corresponding normal displacement on $z = 0$ is given by (9·3) where now

$$\left.\begin{aligned} w_1 &= (a_1^2 - r^2)^{\frac{1}{2}} + \int_{a_1}^{\infty} \frac{t-(t^2-a_1^2)^{\frac{1}{2}}}{(t^2-r^2)^{\frac{1}{2}}}\,\mathrm{d}t \quad (0 \leqslant r \leqslant a_1),\\ w_1 &= \int_r^{\infty} \frac{t-(t^2-a_1^2)^{\frac{1}{2}}}{(t^2-r^2)^{\frac{1}{2}}}\,\mathrm{d}t \quad (a_1 \leqslant r < \infty).\end{aligned}\right\} \tag{9·5}$$

In a similar manner we may deduce general formulae for the half space from those for the crack by defining $f(r)$ and $g(r)$ to be zero outside an interval $0 \leqslant r \leqslant a_1$ where $a_1 \leqslant a^*$, and then making $a^* \to \infty$. For example formula (7·32) then becomes

$$\left.\begin{aligned} H(t) &= \frac{2}{\pi}\int_0^t \frac{rf(r)\,\mathrm{d}r}{(t^2-r^2)^{\frac{1}{2}}} \quad (0 \leqslant t \leqslant a_1),\\ H(t) &= \frac{2}{\pi}\int_0^{a_1} \frac{rf(r)\,\mathrm{d}r}{(t^2-r^2)^{\frac{1}{2}}} \quad (a_1 \leqslant t < \infty),\end{aligned}\right\} \tag{9·6}$$

and $T'(r,z)$ from (7·18) becomes

$$T'(r,z) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{h(t)\,\mathrm{d}t}{\{r^2+(\zeta+\mathrm{i}t)^2\}^{\frac{1}{2}}}. \tag{9·7}$$

Since $f(r)$ is sectionally continuous we see from (9·6) that $H(t) = 0(1/|t|)$, and $h(t) = 0(1/t^2)$, for large $|t|$. Hence (7·23) becomes

$$\left(\frac{r_1}{r_3}\right)^{\frac{1}{2}} \frac{\partial\chi(r,z)}{\partial z} = \frac{1}{2}\int_{-\infty}^{\infty} h(t)\ln[\zeta+\mathrm{i}t+\{r^2+(\zeta+\mathrm{i}t)^2\}^{\frac{1}{2}}]\,\mathrm{d}t. \tag{9·8}$$

Difficulties occur in finding the corresponding values of $\chi(r,z)$ from (7·24). However, we only require derivatives of χ and the x and y derivatives can be found without difficulty. Thus, from (9·8)

$$\left(\frac{r_1}{r_3}\right)^{\frac{1}{2}} \frac{\partial^2\chi(r,z)}{\partial x\,\partial z} = \frac{x}{2}\int_{-\infty}^{\infty} \frac{h(t)\,\mathrm{d}t}{\{r^2+(\zeta+\mathrm{i}t)^2\}^{\frac{1}{2}}[\zeta+\mathrm{i}t+\{r^2+(\zeta+\mathrm{i}t)^2\}^{\frac{1}{2}}]}$$

and†

$$\frac{r_1}{r_3}\frac{\partial\chi(r,z)}{\partial x} = -\frac{x}{2}\int_{-\infty}^{\infty} \frac{h(t)\,\mathrm{d}t}{\zeta+\mathrm{i}t+\{r^2+(\zeta+\mathrm{i}t)^2\}^{\frac{1}{2}}}, \tag{9·9}$$

with a similar expression for $\partial\chi/\partial y$ got by replacing x by y.

† We observe that $\partial\chi(r,z)/\partial x$ satisfies equation (7·12) and suitable conditions at infinity.

Further, (7·29) is replaced by

$$\left.\begin{aligned} q(t) &= \frac{2}{\pi}\int_0^t \frac{r[Xg(r)-Yf(r)]}{(t^2-r^2)^{\frac{1}{2}}}\,\mathrm{d}r \quad (0 \leqslant t \leqslant a_1), \\ q(t) &= \frac{2}{\pi}\int_0^{a_1} \frac{r[Xg(r)-Yf(r)]}{(t^2-r^2)^{\frac{1}{2}}}\,\mathrm{d}r \quad (a_1 \leqslant t < \infty), \end{aligned}\right\} \tag{9·10}$$

so that $q(t) = O(1/|t|)$ for large $|t|$. Finally, the formula corresponding to (7·25) is

$$\frac{\partial\phi}{\partial z} = \frac{1}{2\mathrm{i}}\int_{-\infty}^{\infty} \frac{q(t)\,\mathrm{d}t}{\{r^2+(z+\mathrm{i}t)^2\}^{\frac{1}{2}}} \tag{9·11}$$

and we also find that

$$\frac{\partial\phi}{\partial x} = \frac{x}{2\mathrm{i}}\int_{-\infty}^{\infty} \frac{q(t)\,\mathrm{d}t}{\{r^2+(z+\mathrm{i}t)^2\}^{\frac{1}{2}}[z+\mathrm{i}t+\{r^2+(z+\mathrm{i}t)^2\}^{\frac{1}{2}}]}. \tag{9·12}$$

On inspection we see that we may take the limit of the above solution as $a_1 \to \infty$ provided we stipulate that $f(r) = g(r) = O(1/r^\alpha)$ $(\alpha > 1)$ as $r \to \infty$. With this proviso the integrals in (9·8), (9·9), (9·11) and (9·12) are unchanged in form, although (9·6) and (9·10) now reduce to

$$\left.\begin{aligned} H(t) &= \frac{2}{\pi}\int_0^t \frac{rf(r)}{(t^2-r^2)^{\frac{1}{2}}}\,\mathrm{d}r, \\ q(t) &= \frac{2}{\pi}\int_0^t \frac{r[Xg(r)-Yf(r)]}{(t^2-r^2)^{\frac{1}{2}}}\,\mathrm{d}r, \end{aligned}\right\} \tag{9·13}$$

for all t. Also, from (7·13), or (8·9) in the incompressible case, we see that these conditions are sufficient to ensure that the stresses are zero at infinity.

Thus a solution has been found to a half-space problem in which the shear stresses are zero and the normal pressure $\epsilon g(r)$ and temperature $\epsilon f(r)$ are prescribed arbitrarily on the surface $z = 0$ under the restrictions that $f(r)$, $g(r)$ are sectionally continuous with at most a finite number of finite discontinuities and are of $O(1/r^\alpha)$ $(\alpha > 1)$ as $r \to \infty$.

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